Type Theory v. Geometry

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(jww J. Weinberger & jww S. Awodey)

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Cher André,

We met in 2007, at the Fields Institute. We've been collaborating ever since. The only gap in our collaboration has been 2013, something must have happened that year...

What I owe to you is *immense*. I learn from you category theory. Not so much its results but its practice, and—dare I say—its ethics.

I believe you are one of the few that took Grothendieck's style seriously. You share the same obsession and enthusiasm for discovering and advertising the proper setting to understand a result and to prove it. With you, I experienced the power of category theory not only as a tool for proving, but for understanding.

Another thing that I learn from you, that you drilled into my head, is the importance of examples. So as a tribute, this talk is only going to be about examples!

In this talk, I want to advertise some connections between

Dependent type theory	**Geometry**
contexts	spaces
dependent types	bundles
universes	classifying spaces

I want to sketch their common structure.

Dependencies

Both DTT & geometry make extensive use of Grothendieck fibrations.

$$\begin{array}{c} E \\ \downarrow \\ B \end{array}$$

whose structure we need to find out (cf. Steve's Algebraic Type Theory).

We're gonna look for it through examples.

We're gonna look at quantification within these fibrations,

and at representation of these fibrations by universes.

Dependencies

Index	Families
variables	predicate
context	dependent type
small sets	κ-small sets
small hspaces	κ-small hspaces
simplicial sets	Kan fibrations
top. spaces	open immersions
topoi	étale maps
small categories	left fibrations
manifolds	vector bundles
hspaces	parametrized spectra
schemes	quasi-coherent sheaves
schemes	constructible sheaves
...	...

Quantification

	Index	Family	change index	left image	right image
Predicates	variables	predicates	subst.	$\exists$	$\forall$
DTT	contexts	dep. types	subst.	Σ	Π
CT Fibration $E \to B$	object in base	object in fiber	u^*	$u_!$	u_*

+ compatibility between substitution & left/right images
(= Beck–Chevalley conditions)

Quantification

For any cardinal κ we have a fibration $\begin{array}{c} Set^{\overset{\kappa}{\to}} \\ \big\downarrow p_\kappa \\ Set \end{array}$ whose objects are small families of κ-small sets.

Let $\sigma < \kappa$ such that for any $\beta < \kappa$, $\sigma.\beta < \kappa$.
Then the fibration has α-small sums.

$$B \xrightarrow{\kappa} A \xrightarrow{\alpha} I \quad \mapsto \quad B \xrightarrow{\kappa} I$$

In the language of Steve, we have an action $p_\sigma . p_\kappa \to p_\kappa$
(where $p_\sigma . p_\kappa$ is the polynomial composition).

We put $\Sigma(\kappa) = \sup \sigma$.
If κ is regular, then $\Sigma(\kappa) = \kappa$
and the class of κ-small maps is closed under composition.

Quantification

We can do the same thing for products

Let $\pi < \kappa$ such that for any $\beta < \kappa$, $\beta^\pi < \kappa$.
Then the fibration has π-small products.

$$B \xrightarrow{\kappa} A \xrightarrow{\alpha} I \quad \mapsto \quad \prod_A B \xrightarrow{\kappa} I$$

In the language of Steve, we have an action $p_\pi(p_\kappa) \to p_\kappa$
(where $p_\pi(p_\kappa)$ is the evaluation).

We put $\Pi(\kappa) = \sup \sigma$.
If κ is inaccessible, then $\Pi(\kappa) = \kappa$
and the class of κ-small maps is closed under exponential.

Quantification

This was the elementary version of a more general construction.

Given a Grothendieck fibration $E \to B$, we have the Beck–Chevalley conditions on cartesian square in B

$$\begin{tikzcd} Y' \arrow[r, "v'"] \arrow[d, "u'"'] \arrow[dr, phantom, "\ulcorner" very near start] & Y \arrow[d, "u"] \\ X' \arrow[r, "v"] & X \end{tikzcd} \qquad \begin{tikzcd} E(Y') \arrow[d, shift right=2, "u'_!"'] \arrow[d, shift left=2, "u'_*"] & E(Y) \arrow[l, "(v')^*"'] \arrow[d, shift right=2, "u_!"'] \arrow[d, shift left=2, "u_*"] \\ E(X') \arrow[u] & E(X) \arrow[l, "v^*"] \arrow[u] \end{tikzcd}$$

If $u_!/u'_!$ or u_*/u'_* exist, the left and right Beck–Chevalley conditions are

$$u_! v^* \xrightarrow{\sim} (v')^* u'_! \qquad\qquad v^* u_* \xrightarrow{\sim} u'_* (v')^* .$$

Quantification

The Beck–Chevalley conditions distinguish two classes of maps in the base category B.

A map u is said to be smooth if, for every pullback $u' \to u$, the functor $u_!$ exists and satisfies the left BC cdt.

A map u is said to be proper if, for every pullback $u' \to u$, the functor u_* exists and satisfies the right BC cdt.

The classes of smooth/proper maps are closed under base change.
They define fibrations $\Sigma(E) \to B$ and $\Pi(E) \to B$.

When B is lex, we have the codomain fibration $B^{\to} \to B$,
A subfibration of $B^{\to} \to B$ containing all iso is called a calibration.

$\Sigma(E)$ and $\Pi(E)$ are the maximal calibrations along which sums and products exists in E.

Quantification

Given a calibration $C \subseteq B^{\to} \to B$,

The fibration of C-indexed families in E is

$$\begin{tikzcd}
Fam_C(E) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & E \arrow[d] \\
C \arrow[r, "dom"] \arrow[d, "cod"] & B \\
B &
\end{tikzcd}$$

The fibration $E \to B$ has C-indexed sums (products) if the canonical map

$$\begin{tikzcd}
E \arrow[rr, shift left=3, leftarrow, red] \arrow[rr] \arrow[rr, shift right=3, leftarrow, blue] \arrow[dr] & & Fam_C(E) \arrow[dl] \\
& B &
\end{tikzcd}$$

has a left (right) adjoint fibered over B.
$E \to B$ has sums (products) iff every map in C is left (right) Beck–Chevalley.

Quantification

Fibration $E \to B$

B	E	$\Sigma(E)$	$\Pi(E)$
hspaces	all maps	all maps	all maps
SSet	Kan fibrations	Kan fibrations	? $\supset$ Kan fibrations
Top. sp.	open immersions	open maps	proper maps
1-Topoi	étale maps	locally connected maps (Johnstone)	tidy maps (Moerdijk–Vermeulen)
∞-Topoi	étale maps	locally contractible maps (Martini–Wolf)	Proper maps (Lurie,MW)

Quantification

Fibration $E \to B$

B	E	$\Sigma(E)$	$\Pi(E)$
Categories	left fibrations	smooth functors $\supset$ left fibrations (Grothendieck)	proper functors $\supset$ right fibrations (Grothendieck)
Categories	right fibrations	proper functors (Grothendieck)	smooth functors (Grothendieck)
Categories	all functors	all functors	Conduché fib.
Manifold	Vector bundles	(do not exist)	(do not exist)
hspaces	parametrized spectra	all maps (relative homology)	all maps (relative cohomology)

Universes

For any category B we have an embedding

$$B \subset [B^{\mathrm{op}}, CAT] \; .$$

I call virtual objects of B the objects of $[B^{\mathrm{op}}, CAT]$.

The fibration $E \to B$ defines a virtual object $\mathbb{E} \in [B^{\mathrm{op}}, CAT]$

Any category object in B defines a virtual object (which is then called representable).

A category B has a universe for the fibration E if E is representable (or a filtered union of representable objects).

If B is lex, the codomain fibration defines a virtual object $\mathbb{B} \in [B^{\mathrm{op}}, CAT]$.

If B has a universe $\mathbb{B}$ for $B^{\to} \to B$ then that B is LCC with descent for any existing colimit.

Very few categories have universes (none of the geometric settings).

But they can have universes for other fibrations that $B^{\to} \to B$.

Universes

Often the universe is not representable in B but some approximation exists.

We will see to show how much univalent *and* non-univalent universes are ubiquitous.

Index	Families	Universe (univalent)	Multiverse (multivalent)
small sets	κ-sets	1-groupoid of κ-sets	set of well ordered κ-sets
small hspaces	κ-hspaces	hspaces of κ-hspaces	(no need)
simplicial sets	Kan fibrations	1-stack of Kan fibrations + iso	(add well order)
top. spaces	open immersions	Sierpinski space	(no need) $\{0\} \subset \mathbb{R} \supset \mathbb{R}^*$
1-topoi	étale maps	object classifier	(no need)
locales	étale maps	object classifier	Joyal–Tierney cover (add $\mathbb{N} \twoheadrightarrow X$)

Universes

Index	Families	Universe (univalent)	Multiverse (multivalent)
small categories	left fibrations	2-category of 1-gpd	1-category of 1-gpd with objects
manifolds	vector bundles	$BGL(n)$	$Gr(n, \infty)$ (add embedding into $\mathbb{R}^\infty$)
Category	with families	1-stack of families	Steve's projective cover

Universes

The quantification structure of the fibration must be inherited by the universe (cf. Steve's natural models).

Let's see the example of Topoi.

Let $\mathbf{A}$ be the topos dual to the free logos on 1 generator $Sh(\mathbf{A}) = S[X] = [\mathrm{Fin}, S]$ (object classifier).

Let $\mathbf{A}^\bullet$ be the topos dual to the logos classifying pointed objects $Sh(\mathbf{A}^\bullet) = S[X]_{/X} = [Fin^\bullet, S]$ (where $Fin^\bullet$ is pointed finite spaces).

The projection $\mathbf{A}^\bullet \to \mathbf{A}$ (forgetting the base point) is the universal étale map, representing the fibration of étale maps.

	Σ	Π	Id
$\mathbf{A}^\bullet \to \mathbf{A}$	yes (composition of étale maps)	only along proper + etale map = finite object	yes (diagonal étale map is étale)

Universes

Why no Π-types ?

$Sh(\mathbf{X})$ is known to have Π-types!

But they are not preserved by geometric morphisms.

This is a failure of Beck–Chevalley condition:

$$
\begin{tikzcd}
\mathbf{Y}' \arrow[r, "v'"] \arrow[d, "u'"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbf{Y} \arrow[d, "u"] \\
\mathbf{X}' \arrow[r, "v"] & \mathbf{X}
\end{tikzcd}
\qquad
\begin{tikzcd}
Sh(\mathbf{Y}') \arrow[d, shift right, "u_*"] \arrow[dr, phantom, "\ulcorner", very near start] & Sh(\mathbf{Y}) \arrow[l, "(v')^*"'] \arrow[d, shift right, "u_*"] \\
Sh(\mathbf{X}') \arrow[u, shift right] & Sh(\mathbf{X}) \arrow[l, "v^*"'] \arrow[u, shift right]
\end{tikzcd}
$$

$$v^* u_* \xrightarrow{\not\cong} u'_*(v')^*$$

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Punchline

In classical DTT the universe stands alone.

In geometric settings, the universe of $E \to B$ interacts with other objects, necessary to explicit its structure:

1. the codomain fibration of B (which is almost never representable)
2. the calibrations of Σ and Π

From there, there are several approaches

1. embed the geometric setting in a larger complete DTT setting (cohesion, simplicial type theory, synthetic algebraic geometry...)
2. or adapt/generalize the definition of a DTT so that the geometric settings become examples .

I'm very enthusiast about the second approach.

To be continued...

Happy 80th Birthday André!