Eckmann-Hilton and the Hopf Fibration
in Homotopy Type Theory

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The Plan

Use the Eckmann-Hilton argument to construct the Hopf fibration and, consequently, a generator of $\Omega^3(\mathbb{S}^2)$:

1. Use Eckmann-Hilton to construct a 3-loop $\mathrm{eh} : \Omega^3(\mathbb{S}^2)$. This immediately defines a map $\mathrm{hpf} : \mathbb{S}^3 \to \mathbb{S}^2$.
2. Construct a type family $\mathcal{H} : \mathbb{S}^2 \to U$ with $\mathcal{H}$ defined to be $\mathbb{S}^1$ over the base point of $\mathbb{S}^2$.
3. Construct a fiberwise equivalence $(x : \mathbb{S}^2) \to \mathrm{fib}_{\mathrm{hpf}}(x) \simeq \mathcal{H}(x)$

Conventions

$\mathbb{S}^1$

$b_1 : \mathbb{S}^1$
$\mathsf{loop} : \Omega(\mathbb{S}^1)$

$\mathbb{S}^2$

$N_2 : \mathbb{S}^2$
$\mathsf{surf}_2 : \Omega^2(\mathbb{S}^2)$

$\mathbb{S}^3$

$N_3 : \mathbb{S}^3$
$\mathsf{surf}_3 : \Omega^3(\mathbb{S}^3)$

Also I use 1 for the reflexivity path

The Eckmann-Hilton Argument

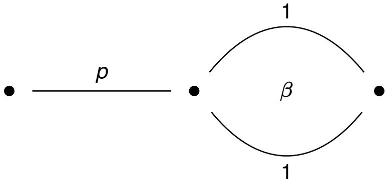
Eckmann-Hilton

For $\beta, \alpha : \Omega^2(X)$, we have $\mathsf{EH}(\beta, \alpha) : \beta \cdot \alpha = \alpha \cdot \beta$

Where does Eckmann-Hilton come from?

For a 2-loop $\beta : \Omega^2(X, \bullet)$, we have a homotopy of type $\mathrm{id}_{\Omega(X)} \sim \mathrm{id}_{\Omega(X)}$ given by the formula:

$$\mathrm{Whisker}_\beta :\equiv \lambda(p).1_p \star \beta$$



Eckmann-Hilton is (more or less) the *naturality condition* of this homotopy when *applied to 2-loops*.

Where does Eckmann-Hilton come from?

This homotopy has a naturality condition induced by paths in $\Omega(X)$. In particular, for $\alpha : \Omega^2(X)$:

$$\begin{array}{ccc}
1^2_\bullet & \overset{(\alpha \star 1^2_\bullet)}{\text{———————}} & 1^2_\bullet \\
{\scriptstyle (1^2_\bullet \star \beta)} \Big| & \text{nat-Whisker}_\beta(\alpha) & \Big| {\scriptstyle (1^2_\bullet \star \beta)} \\
1^2_\bullet & \underset{(\alpha \star 1^2_\bullet)}{\text{———————}} & 1^2_\bullet
\end{array}$$

Plus coherence paths, this lends

$$\text{EH}(\beta, \alpha) : \beta \cdot \alpha = \alpha \cdot \beta$$

.

The Eckmann-Hilton 3-loop

We can use this to construct a 3-loop in $\mathbb{S}^2$:

$$1^2_{\mathsf{N}_2} \;\text{———}\; \mathsf{surf}_2 \cdot \mathsf{surf}_2^{-1} \;\xrightarrow{\;\mathsf{EH}(\mathsf{surf}_2,\mathsf{surf}_2^{-1})\;}\; \mathsf{surf}_2^{-1} \cdot \mathsf{surf}_2 \;\text{———}\; 1^2_{\mathsf{N}_2}$$

Call this loop eh.

The map hpf

The 3-loop eh lets us define "the Hopf fibration":

hpf : $\mathbb{S}^3 \to \mathbb{S}^2$

Define a map hpf : $\mathbb{S}^3 \to \mathbb{S}^2$ by $\mathbb{S}^3$-induction:

$$\mathrm{hpf}(\mathrm{N}_3) :\equiv \mathrm{N}_2$$
$$\mathrm{hpf}(\mathrm{surf}_3) := \mathrm{surf}_2.$$

The Plan, part II

Now that we have a map $\mathbb{S}^3 \to \mathbb{S}^2$, we need to show this map is* the Hopf fibration. How are we going to do this?

1. Construct $\mathcal{H} : \mathbb{S}^2 \to U$ with $\mathcal{H}(\mathsf{N}_2) :\equiv \mathbb{S}^1$
2. Construct $(x : \mathbb{S}^2) \to \mathsf{fib}_{\mathsf{hpf}}(x) \simeq \mathcal{H}(x)$

*up to sign, though I'm pretty sure it *is* the Hopf fibration. I can say more about this at the end.

Interlude, descent data of $\mathbb{S}^2$

Descent on $\mathbb{S}^2$

A type family B over $\mathbb{S}^2$ is equivalent to:

a type X, the value of $B(\mathsf{N}_2)$

a homotopy $\mathrm{id}_X \sim \mathrm{id}_X$, the value of $B(\mathsf{surf}_2)$ (together with univalence)

The equivalence is given by $(B : \mathbb{S}^2 \to U) \mapsto (B(\mathsf{N}_2), \mathrm{tr}^{(B)^2}(\mathsf{surf}_2))$

Selecting the Family $\mathcal{H}$, some motivation

The descent data of $\mathrm{fib}_{\mathrm{hpf}}$

The type: $\mathrm{fib}_{\mathrm{hpf}}(\mathrm{N}_2)$

The homotopy: $\lambda(z,p).(1_z, 1_p \star \mathrm{surf}_2) \quad \equiv \lambda(z,p).(1_z, \mathrm{Whisker}_\beta(p))$

Selecting the Family $\mathcal{H}$, some motivation

The family $(x : \mathbb{S}^2) \to \mathrm{fib}_{\mathrm{hpf}}(x)$ comes equipped with a section

$$(z : \mathbb{S}^3) \to \mathrm{fib}_{\mathrm{hpf}}(\mathrm{hpf}(z))$$

given by

$$\lambda(z).(z, 1_{\mathrm{hpf}(z)})$$

For any $B : \mathbb{S}^2 \to U$, fiberwise maps $(x : \mathbb{S}^2) \to (\mathrm{fib}_{\mathrm{hpf}}(x) \to B(x))$ are equivalent to sections $(z : \mathbb{S}^3) \to B \circ \mathrm{hpf}(z)$

Universal Property of $\mathrm{fib}_{\mathrm{hpf}}$

$\mathrm{fib}_{\mathrm{hpf}}$ is initial among type families $B : \mathbb{S}^2 \to U$ equipped with a section $(z : \mathbb{S}^3) \to B \circ \mathrm{hpf}(z)$.

Selecting the Family $\mathcal{H}$, some motivation

What does a section $(z : \mathbb{S}^3) \to B \circ \mathrm{hpf}(z)$ mean? Such a section is equivalent to:

A point $b : B(\mathrm{N}_2)$
A dependent 3-loop over eh. This means $\mathrm{tr}^{(B)^3}(\mathrm{eh})(b) = 1_b^2$.

But what is $\mathrm{tr}^{(B)^3}(\mathrm{eh})$?

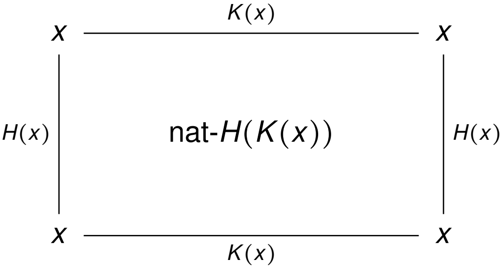
Eckmann-Hilton in the Universe

Given a type $X : U$ and $H, K : \mathrm{id}_X \sim \mathrm{id}_X$, there is a canonical term of type

$$H \cdot_h K \sim K \cdot H$$

given by the formula:

$$\lambda(x).\text{nat-}H(K(x))$$



What is $\mathrm{tr}^{(B)^3}(\mathrm{eh})$?

I claim $\mathrm{tr}^{(B)^3}(\mathrm{eh})$ is this construction repeated for $\mathrm{tr}^{(B)^2}(\mathrm{surf}_2)$ and $\mathrm{tr}^{(B)^2}(\mathrm{surf}_2^{-1})$

$$1_x \;—\; \mathrm{tr}^{(B)^2}(\mathrm{surf}_2)(x) \cdot \mathrm{tr}^{(B)^2}(\mathrm{surf}_2^{-1})(x) \;—\; \mathrm{tr}^{(B)^2}(\mathrm{surf}_2^{-1})(x) \cdot \mathrm{tr}^{(B)^2}(\mathrm{surf}_2)(x) \;—\; 1_x$$

Thus, if $\mathrm{tr}^{(B)^3}(\mathrm{eh})(b)$ is trivial, it must be that

$$\text{nat-}[\mathrm{tr}^{(B)^2}(\mathrm{surf}_2)](\mathrm{tr}^{(B)^2}(\mathrm{surf}_2^{-1})(b))$$

is trivial. I.e., is equal to $\text{r-inv} \cdot \text{l-inv}^{-1}$ (more or less)

A(n informal) Characterizaton of $\mathrm{fib}_{\mathrm{hpf}}$

The family $\mathrm{fib}_{\mathrm{hpf}}$

The initial type family over $\mathbb{S}^2$ equipped with a point in $\mathrm{fib}_{\mathrm{hpf}}(\mathrm{N}_2)$ such that the naturality condition of the 2-dimensional descent data is trivial.

The type family $\mathcal{H}$, a definition

the family $\mathcal{H}$

Define $\mathcal{H}$ with the following descent data:
$\mathbb{S}^1$ and L^{-1}

Where $\mathsf{L}^{-1} : \mathsf{id}_{\mathbb{S}^1} \sim \mathsf{id}_{\mathbb{S}^1}$ is defined by:

$\mathsf{L}^{-1}(\mathsf{b}_1) \equiv \mathsf{loop}^{-1}$ and

$\text{nat-}\mathsf{L}^{-1}(\mathsf{b}_1) \equiv \text{l-inv}_{\mathsf{loop}} \cdot \text{r-inv}_{\mathsf{loop}}$

which has type $\mathsf{loop}^{-1} \cdot \mathsf{loop} = \mathsf{loop} \cdot \mathsf{loop}^{-1}$.

The Section for $\mathcal{H}$

The section $(z : \mathbb{S}^3) \to \mathcal{H} \circ \mathsf{hpf}(z)$

We have a section $\mathsf{diag} : (z : \mathbb{S}^3) \to \mathcal{H} \circ \mathsf{hpf}(z)$.

We can define this by $\mathbb{S}^3$-induction:

Set $\mathsf{diag}(\mathsf{N}_3) :\equiv \mathsf{b}_1$.

Need a dependent 3-loop over surf_3 at b_1. This amounts to a 3-path in $\mathbb{S}^1$, a 1-type. We get this for free :)

Constructing the equivalence

We want an equivalence $(x : \mathbb{S}^2) \to (\mathrm{fib}_{\mathrm{hpf}}(x) \to \mathcal{H}(x))$. We do this as follows:

1. Construct a forwards map $f : (x : \mathbb{S}^2) \to (\mathrm{fib}_{\mathrm{hpf}}(x) \to \mathcal{H}(x))$
2. Construct a backwards map $g : (x : \mathbb{S}^2) \to (\mathcal{H}(x) \to \mathrm{fib}_{\mathrm{hpf}}(x))$
3. Construct a fiberwise homotopy $(x : \mathbb{S}^2) \to (f_x \circ g_x \sim \mathrm{id}_{\mathcal{H}(x)})$
4. Construct a fiberwise homotopy $(x : \mathbb{S}^2) \to (g_x \circ f_x \sim \mathrm{id}_{\mathrm{fib}_{\mathrm{hpf}}(x)})$

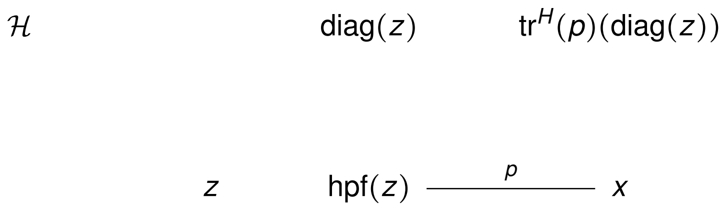
The Forwards Map $f : (x : \mathbb{S}^2) \to (\mathrm{fib}_{\mathrm{hpf}}(x) \to \mathcal{H}(x))$

The Forwards Map

We have a map $f : (x : \mathbb{S}^2) \to (\mathrm{fib}_{\mathrm{hpf}}(x) \to \mathcal{H}(x))$

We can define this as:

$$f_x(z, p) :\equiv \mathrm{tr}^H(p)(\mathrm{diag}(z))$$



The Backwards Map $g : (x : \mathbb{S}^2) \to (\mathcal{H}(x) \to \mathrm{fib}_{\mathrm{hpf}}(x))$

By $\mathbb{S}^2$-induction...

The data of such a map

1. A map $g_{\mathrm{N}_2} : \mathbb{S}^1 \to \mathrm{fib}_{\mathrm{hpf}}(\mathrm{N}_2)$
2. A homotopy $[\mathrm{tr}^{(\mathrm{fib}_{\mathrm{hpf}})^2}(\mathrm{surf}_2)] \cdot_r g_{\mathrm{N}_2} \sim g_{\mathrm{N}_2} \cdot_l \mathrm{L}^{-1}$

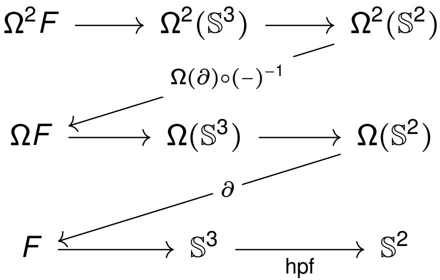
equivalently...

1. a point $a : \mathrm{fib}_{\mathrm{hpf}}(\mathrm{N}_2)$
2. loop $q : \Omega(\mathrm{fib}_{\mathrm{hpf}}(\mathrm{N}_2))$ at a
3. a path $\mathrm{tr}^{(\mathrm{fib}_{\mathrm{hpf}})^2}(\mathrm{surf}_2)(a) = q^{-1}$ – can make this path 1
4. a naturality square... showing that the naturality conditions of the two homotopies are the same

Item 4 essentially amounts to showing that
$\mathrm{nat}\text{-}[\mathrm{tr}^{(\mathrm{fib}_{\mathrm{hpf}})^2}(\mathrm{surf}_2)](q) = \text{r-inv} \cdot \text{l-inv}^{-1}$

Choosing the data

1. a point $a : \mathrm{fib}_{\mathrm{hpf}}(\mathsf{N}_2)$
2. loop $q : \Omega(\mathrm{fib}_{\mathrm{hpf}}(\mathsf{N}_2))$ at a



Choosing the naturality square

We can write out the boundary of this square... (I won't now, since this involves a few tedious computations to determine the boundary)

The type of fillers ends up being equivalent to

$$\mathrm{fib}_{\mathrm{hpf}}(\mathrm{eh})$$

use $(\mathrm{surf}_3, 1)$

The homotopy $(x : \mathbb{S}^2) \to f_x \circ g_x \sim \mathrm{id}_{H(x)}$

This is easy

The homotopy $(x : \mathbb{S}^2) \to g_x \circ f_x \sim \mathrm{id}_{H(x)}$

$(x : \mathbb{S}^2)(w : \mathrm{fib}_{\mathrm{hpf}}(x)) \to g_x \circ f_x(w) = w$

Not so easy... But...

Such a homotopy is equivalent to

$$(z : \mathbb{S}^3) \to g_{\mathrm{hpf}(z)}(\mathrm{diag}(z)) = (z, 1_{\mathrm{hpf}(z)})$$

(more or less) requires us to show that $g_{\mathrm{eh}}(\mathrm{b}_1) = (\mathrm{surf}_3, 1)$

(more or less) amounts to showing we used $(\mathrm{surf}_3, 1)$ to fill the naturality square which defined g

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The End :)

Questions? Comments?