Formalization & Computation: Categorical Normalization by Evaluation

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Outline

Categorical Normalization

1. Normalization (by Evaluation)
2. Naïve Categorical Normalization
3. P-Category Theory & P-Categorical Normalization
4. Correctness

Coq Formalization

1. Design Decisions
2. Basic Constructions
3. Cartesian Structures
4. (Co)Ends & Presheaf Exponential

Why Normalization?

Problem

Decide for STLC.

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- Clearly decidable.

Algorithmic Problem

How to compute?

$$\Gamma \vdash t : T \rightsquigarrow \Gamma \vdash \mathsf{nf}(t) : T$$

Normalization by Evaluation

The standard approach after Berger and Schwichtenberg '91 proceeds as follows.

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1. Define neutral, $\mathcal{M}_{T,\Gamma}$, and normal, $\mathcal{N}_{T,\Gamma}$, terms, as subsets of all terms, $\mathcal{L}_{T,\Gamma}$.
2. Define a particular model for types, $[\![T]\!]_\Gamma$.
3. Interpret terms into the model, $\mathcal{L}_{T,\Gamma} \to [\![T]\!]_\Gamma$.
4. Define maps
   - $q : \mathcal{M}_{T,\Gamma} \to [\![T]\!]_\Gamma$
   - $u : [\![T]\!]_\Gamma \to \mathcal{N}_{T,\Gamma}$
5. Define $\mathsf{nf} : \mathcal{L}_{T,\Gamma} \to \mathcal{N}_{T,\Gamma}$

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Mathematical/Categorical Justification?

From where does all this come?

A.H.S. ‘95 provides some categorical justification, using an *ad-hoc* gluing-style argument.

Č.D.S. ‘98 uses an alternative categorical foundation.

Fiore ‘02 provides a fully categorical foundation using gluing.

Naïve Categorical Normalization (I)

Generic Interpretation

For any Cartesian-closed category, $\mathbb{M}$, there is a universal Cartesian-closed interpretation functor, $[\![-]\!]$, from the free Cartesian-closed category, $\mathcal{F}$, over a basetype:

$$\mathcal{F} \xrightarrow{[\![-]\!]} \mathbb{M}$$

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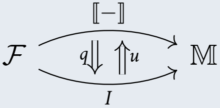


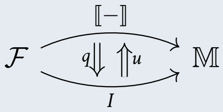
I-Normalization

A normalization function for some model, $\mathbb{M}$, and interpretation functor, $I : \mathcal{F} \to \mathbb{M}$, can be constructed for $\sigma : \mathcal{F}(\Gamma, \Delta)$:

- $u_\Gamma : I(\Gamma) \to [\![\Gamma]\!]$
- $[\![\sigma]\!] : [\![\Gamma]\!] \to [\![\Delta]\!]$
- $q_\Delta : [\![\Delta]\!] \to I(\Delta)$

$\Big\}$ $\mathsf{nf}_I(\sigma) : I(\Gamma) \to I(\Delta)$

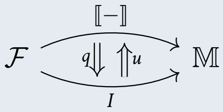
Choice of $\mathbb{M}$ and I



$\mathbb{M}$	I	Utile? (✓/ ✗)

Naïve Categorical Normalization (II)

Choice of $\mathbb{M}$ and I



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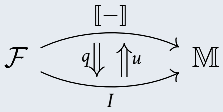

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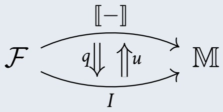

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Problem

In fact, no matter what category we choose for our model all normalization functions will be inutile as they are all extensionally the identity. Following Č.D.S. we switch to a more intensional setting: P-category theory.

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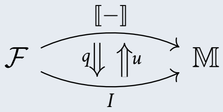

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Silver Lining

So far the standard category theory has created a framework for normalization which has avoided defining neutral and normal forms.

Warning!

Nota Bene

In the sequel I use some terminology not in its precise HoTT/UF sense!

P-Category Theory (I) [Č.D.S. '98]

Partial Equivalence Relation (PER)

A relation which is symmetric and transitive.

P-Set

A collection with a given PER.
We denote the underlying collection of a P-set, X, by $|X|$.
We denote the associated PER by $\sim_X$.

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Intuition

Think of $(|X|, \sim_X) \simeq \{x : X \mid x \sim x\}/\sim_X$.

This allows simultaneously taking a subset and a quotient.

This provides (co)completeness properties.

P-Category Theory (II)

P-Category

A P-category is given by the following:

- a collection of objects;
- a P-set of arrows between objects;
- a composition operation for arrows; and
- an identity arrow;

} DATA

such that:

- $f \sim f' \wedge g \sim g' \Rightarrow f \circ g \sim f' \circ g'$;
- $f \sim f' \wedge g \sim g' \wedge h \sim h' \Rightarrow (f \circ g) \circ h \sim f' \circ (g' \circ h')$;
- $\mathsf{id}_x \sim \mathsf{id}_x$;
- $f \sim f' \Rightarrow \mathsf{id}_x \circ f \sim f'$; and
- $f \sim f' \Rightarrow f \circ \mathsf{id}_x \sim f'$.

} AXIOMS

P-Category Theory (III)

P-Functor

A P-functor, F, from P-category, $\mathbb{C}$, to P-category, $\mathbb{D}$, is given by the following:

- a map of objects; and
- a P-map of arrows between objects;

} DATA

such that:

- $f \sim f' \Rightarrow F\,f \sim F\,f'$ (this is the P-map condition);
- $f \sim f' \wedge g \sim g' \Rightarrow F\,(f \circ g) \sim F\,f' \circ F\,g'$; and
- $F\,\mathsf{id}_x \sim \mathsf{id}_{F\,x}$.

} AXIOMS

P-Category Theory (IV)

P-Functor Category

The P-functor category, $[\mathbb{C}, \mathbb{D}]$, has:

- P-functors, $\mathbb{C} \to \mathbb{D}$, as objects; and
- *all* transformations as morphisms, where $\alpha \sim \beta$ when:
  - α is P-natural;
  - β is P-natural; and
  - $\alpha_x \sim \beta_x$, for all x.

P-Naturality

α is P-natural when:

$$f \sim f' \Rightarrow (\alpha_y \circ F f) \sim (F f' \circ \alpha_x)$$

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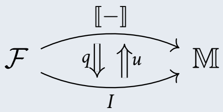
Observation

Note that the PER for the morphisms in functor categories typifies the P-categorical approach of taking subsets, by predicating both α and β.

P-Categorical Normalization

Choice of $\mathbb{M}$ and I



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P-Categorical Normalization

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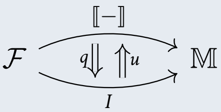

$\mathbb{M}$	I	Utile? (✓/ ✗)
$\mathcal{F}$	Id	✗
$\widehat{\mathcal{F}}$	よ	✓

Success!

By switching into the intensional P-categorical setting we can elucidate the intensional behaviour separately from the extensional properties. We now have a putative computational algorithm: only thing we have to do now is formalize it ...

Correctness

Correctness Properties

Correctness of normalization algorithms arises in these properties:

- $t \equiv_{\beta\eta} t' \Rightarrow \mathsf{nf}(t) \equiv_{\alpha} \mathsf{nf}(t')$;
- $t \equiv_{\beta\eta} \mathsf{nf}(t)$;
- $\mathsf{nf}(t) \in \mathcal{N}$; and
- $t \in \mathcal{N} \Rightarrow t \equiv_{\alpha} \mathsf{nf}(t)$.

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- $t \in \mathcal{N} \Rightarrow t \equiv_{\alpha} \mathsf{nf}(t)$.

Properties for Free

The P-categorical construction give us the following for free:

- $t \equiv_{\beta\eta} t' \Rightarrow \mathsf{nf}(t) \equiv_{\beta\eta} \mathsf{nf}(t')$; and
- $t \equiv_{\beta\eta} \mathsf{nf}(t)$.

Correctness Proof (Sketch) (I)

Category of Renamings

The category of contexts and context renamings, $\mathcal{R}$, is a subcategory of $\mathcal{F}$ with inclusion functor i.

Presheaves of Neutral and Normal Terms

Neutrality and Normality are preserved under renamings, allowing them to lift to type-indexed families of presheaves.

$$\mathcal{M}, \mathcal{N} : \mathsf{Ty} \to \widehat{\mathcal{R}}$$

Gluing Category

The gluing category, $\mathcal{G} \triangleq \widehat{\mathcal{R}}{\downarrow}i^*$, is Cartesian-closed. Furthermore, the codomain projection functor is Cartesian-closed.

Correctness Proof (Sketch) (II)

Interpretation in $\mathcal{G}$

The interpretation in $\mathcal{G}$ is induced by:

$$\mathcal{G}[\![\iota]\!] \triangleq \begin{array}{c} \mathcal{M}_\iota \\ \downarrow \\ i^*\left(\widehat{\mathcal{F}}[\![\iota]\!]\right) \end{array}$$

We denote the domain presheaf in $\widehat{R}$ by $\mathcal{I}$.

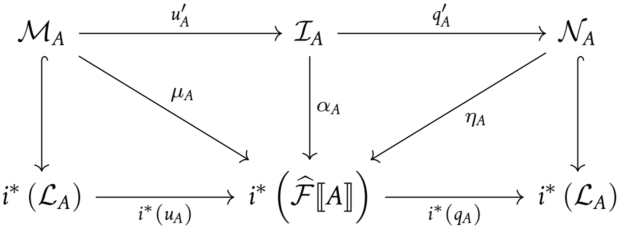
Diagram in $\mathcal{G}$

We define the following type-indexed diagram in $\mathcal{G}$ by induction on A:

$$
\left(\begin{array}{c} \mathcal{M}_A \\ \Big\downarrow \mu_A \\ i^*\left(\widehat{\mathcal{F}}[\![A]\!]\right) \end{array} \right)
\xrightarrow{\mathsf{in}_A}
\left(\begin{array}{c} \mathcal{I}_A \\ \Big\downarrow \alpha_A \\ i^*\left(\widehat{\mathcal{F}}[\![A]\!]\right) \end{array} \right)
\xrightarrow{\mathsf{out}_A}
\left(\begin{array}{c} \mathcal{N}_A \\ \Big\downarrow \eta_A \\ i^*\left(\widehat{\mathcal{F}}[\![A]\!]\right) \end{array} \right)
$$

Correctness Proof (Sketch) (IV)

The following is induced:



Coq Formalization

Design Decisions

- Universe Polymorphism
- Cumulative Records
- Yoneda-Centric Definitions

Coq Formalization: PERs & P-Types

PER

```
Cumulative Record PER@{+i +j} (A : Type@{i}) := Build_PER {
  PER_rel : A -> A -> Type@{j};
  PER_symm : forall {x y}, PER_rel x y -> PER_rel y x;
  PER_trans : forall {x y z}, PER_rel x y -> PER_rel y z -> PER_rel x z;
}.
```

P-Type

```
Cumulative Record PType@{+i +j} : Type := Build_PType {
  PType_type :> Type@{i};
  PType_per :> PER@{i j} PType_type;
}.
```

Coq Formalization: P-Category

P-Category

```
Cumulative Record PCat@{+i +j +k} := Build_PCat {
  PCat_obj :> Type@{i};
  PCat_hom : PCat_obj -> PCat_obj -> PType@{j k};
  PCat_id_mor : forall x, PCat_hom x x;
  PCat_comp : forall {x y z}, PCat_hom y z -> PCat_hom x y -> PCat_hom x z;

  PCat_id_rel : forall x, (PCat_id_mor x) ~ (PCat_id_mor x);
  PCat_comp_rel : forall {x y z f f' g g'},
    f ~ f' -> g ~ g' -> (PCat_comp f g) ~ (PCat_comp f' g');
  ...
}.
```

Coq Formalization: P-Cartesian Structures (I)

P-Terminal Objects

```
Definition IsPTermObj {C : PCat} (term : C) :=
  PNatIso
    (PBiFunPartialRight (@PHomFun C) term)
    (PCompFun (PConstFun (C:=PSet) PUnit) PTermFun).
```

$$\mathsf{IsTerminal}(t) \triangleq \mathsf{Hom}_{\mathbb{C}}(-, t) \cong \Delta_{\{*\}}$$

Coq Formalization: P-Cartesian Structures (II)

P-Cartesian Products

```
Definition IsPCartProd {C : PCat} (prod : C -> C -> C) :=
  forall a b,
    PNatIso
      (PBiFunPartialRight PHomFun (prod a b))
      (PCompFun
        (PBiFunPartialRight (PHomFun (C:=PProdCat C C)) (a, b))
        (POppFun (PPairFun PIdFun PIdFun))
      ).
```

$$\mathsf{IsProduct}(- \times =) \triangleq \prod_{a,b\,:\,\mathbb{C}} \mathsf{Hom}_{\mathbb{C}}(\equiv, a \times b) \cong \mathsf{Hom}_{\mathbb{C}\times\mathbb{C}}((\equiv,\equiv),(a,b))$$

Coq Formalization: P-Cartesian Structures (III)

P-Cartesian Exponentials

```
Definition IsPCartExp {C : PCartCat} (exp : C -> C -> C) :=
  forall a b,
    PNatIso
      (PBiFunPartialRight PHomFun (exp b a))
      (PCompFun
        PHomFun
        (PPairFun
          (POppFun (PBiFunPartialRight PCartProdFun a))
          (PCompFun (PConstFun b) PTermFun))
      ).
```

$$\mathsf{IsExponential}(- \Rightarrow =) \triangleq \prod_{a,b\,:\,\mathbb{C}} \mathsf{Hom}_{\mathbb{C}}(\equiv, a \Rightarrow b) \cong \mathsf{Hom}_{\mathbb{C}\times\mathbb{C}}(\equiv \times a, b)$$

P-Set–Valued P-Ends

P-Ends

For $F : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathsf{PSet}$ we have:

- $\left| \int_{c:\mathbb{C}} F(c,c) \right| \triangleq \prod_{c:\mathbb{C}} F(c,c)$
- $w \sim w' \triangleq$
    - $\prod_{x,y:\mathbb{C}} \prod_{f,f':x \to y} f \sim f' \Rightarrow F\,(f, \mathsf{id})\,(w\,y) \sim F\,(\mathsf{id}, f')\,(w\,x) \quad \wedge$
    - $\prod_{x,y:\mathbb{C}} \prod_{f,f':x \to y} f \sim f' \Rightarrow F\,(f, \mathsf{id})\,(w'\,y) \sim F\,(\mathsf{id}, f')\,(w'\,x) \quad \wedge$
    - $\prod_{z:\mathbb{C}} w\,z \sim w'\,z$

P-Set–Valued P-Coends

P-Coends

For $F : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathsf{PSet}$ we have:

- $\left| \int^{c:\mathbb{C}} F(c, c) \right| \triangleq \sum_{c:\mathbb{C}} F(c, c)$
- $w \sim w'$ is inductively generated by the following:
  - $\prod_{z:\mathbb{C}} \prod_{s,s':F\,(z,z)} s \sim s' \Rightarrow (z; s) \sim (z; s')$
  - $\prod_{x,y:\mathbb{C}} \prod_{f,f':y \to x} \prod_{s,s':F\,(x,y)} f \sim f' \Rightarrow s \sim s' \Rightarrow (y; F\,(f, \mathsf{id})\, s) \sim (x; F\,(\mathsf{id}, f')\, s')$
  - $\prod_{x,y:\mathbb{C}} \prod_{f,f':y \to x} \prod_{s,s':F\,(x,y)} f \sim f' \Rightarrow s \sim s' \Rightarrow (x; F\,(\mathsf{id}, f)\, s) \sim (y; F\,(f', \mathsf{id})\, s')$
  - $w_1 \sim w_2 \wedge w_2 \sim w_3 \Rightarrow w_1 \sim w_3$

P-Set–Valued P-(Co)Ends

Properties

- Density Formula for coends.
- Fubini rule for ends.
- Functor Category homs as ends.
- Cocontinuity and Continuity of the Hom-functor.
- Isomorphism under duality of $\mathbb{C}$.

Presheaf Exponential

$$\widehat{\mathbb{C}}\,(K, G^F) \cong \int_c \mathbf{Set}\,(Kc, G^F c) \equiv \int_c \mathbf{Set}\left(Kc, \int_{c'} \mathbb{C}(c', c) \Rightarrow Fc' \Rightarrow Gc'\right)$$
$$\cong \int_c \int_{c'} \mathbf{Set}\Big(Kc, \mathbb{C}(c', c) \Rightarrow Fc' \Rightarrow Gc'\Big)$$
$$\cong \int_{c'} \int_{c} \mathbf{Set}\Big(Kc, \mathbb{C}(c', c) \Rightarrow Fc' \Rightarrow Gc'\Big)$$
$$\cong \int_{c'} \int_{c} \mathbf{Set}\big(Kc \times \mathbb{C}(c', c), Fc' \Rightarrow Gc'\big)$$
$$\cong \int_{c'} \mathbf{Set}\left(\int^{c} Kc \times \mathbb{C}(c', c), Fc' \Rightarrow Gc'\right)$$
$$\cong \int_{c'} \mathbf{Set}(Kc', Fc' \Rightarrow Gc')$$
$$\cong \int_{c'} \mathbf{Set}(Kc' \times Fc', Gc') \cong \widehat{\mathbb{C}}(K \times F, G)$$

Future Work

- Complete formalization of gluing construction
- Move to P-bicategory theory for two-dimensional simple type theory
- Find connections with other categorical/mathematical systems
- Monoidal setting with Day convolution

Any Questions?