Finite colimits in an elementary higher topos
Revised version

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Outline

Elementary topoi

Definition (Elementary 1-topos)

An elementary topos **E** is a category that

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- admits a subobject classifier
- admits enough “universes” (a.k.a object classifiers)

Leaning toward an internal logic

- Finite limits : substitution is modelled by pullback
- Dependent product : Π-types
- Finite colimits : some inductive types ($+$-types, quotients)
- Subobject classifier : inbuilt "predicate logic"
- Object classifiers : type-theoretic universes (inbuilt "untruncated logic")

Finite colimits

In the 1-categorical setting, the existence of finite colimits follows from the other three axioms.

This essentially amounts to the fact that quotient can be defined by "comprehension".

Precisely, the direct proof by Mikkelsen:

Proof.

Given an equivalence relation $R \subset X \times X$, take the image of $X \to P(X)$ "mapping" $x \in X$ to its equivalence class. If R is not an equivalence relation, replace it by the equivalence relation it generates. $\square$

Outline

Monadicity of the powerobject endofunctor

Another proof, due to Paré, reformulates this idea in a more abstract way:

Proposition

$\mathbf{E}^{op}$ *is monadic over* $\mathbf{E}$ *via the powerobject functor:*

$$P : \mathbf{E}^{op} \to \mathbf{E}$$
$$X \mapsto P(X) := \Omega^X$$

Proof.

Check the Beck monadicity theorem hypothesis are verified (notably, P is conservative).

□

The underlying intuition

The "universe of monomorphisms" (i.e the subobject classifier) captures the full logic of **E**:

$$\begin{array}{ccc} S & \longrightarrow & 1 \\ \big\downarrow & \lrcorner & \big\downarrow \\ X & \longrightarrow & \Omega \end{array}$$

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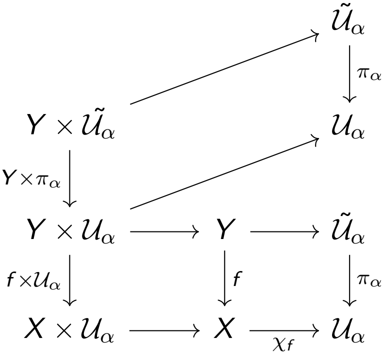
- External natural numbers are not enough:
  - One should consider the free groupoid object on two parallel morphisms.
  - The higher Beck monadicity theorem involves infinite colimits in $\mathcal{E}^{op}$.

Local universe functor

Given an α-small morphism $f : Y \to X$ in $\mathcal{E}$, form the following diagram:

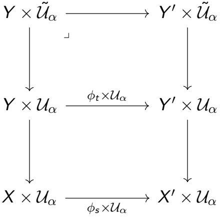


then map f to $(f \times \mathcal{U}_\alpha)_* Y \times \pi_\alpha$.

Local universe functor

The mate transformation gives the action on edges, which is contravariant in the domain:

$$
\begin{aligned}
&P_\alpha f (= f \times \mathcal{U}_\alpha)_* Y \times \pi_\alpha \\
&= (f \times \mathcal{U}_\alpha)_* (\phi_s \times \mathcal{U}_\alpha)^* Y' \times \pi_\alpha) \\
&\uparrow \textbf{(mate)} \\
&(\phi_t \times \mathcal{U}_\alpha)^* P_\alpha f' (= \\
&(\phi_s \times \mathcal{U}_\alpha)^* (f' \times \mathcal{U}_\alpha)_* Y' \times \pi_\alpha)
\end{aligned}
$$



Local universe functor

Write $\mathcal{E}_\alpha^\to$ for the full subcategory of α-small morphisms.
Also write $\mathbf{Poly}(\mathcal{E}_\alpha)$ for the full subcategory of polynomial functors (of the form $1 \leftarrow Y \to X \to 1$) whose underlying map is α-small.

Proposition

$$P_\alpha : \mathcal{E}_\alpha^\to \to \mathbf{Poly}(\mathcal{E}_\alpha)$$

and the analogous functor

$$P'_\alpha : \mathbf{Poly}(\mathcal{E}_\alpha) \to \mathcal{E}_\alpha^\to$$

form an adjoint pair $P_\alpha \dashv P'_\alpha$.

Algebraic simplicial diagrams

We define algebraic simplicial diagrams to be the diagrams

$$\cdots \longrightarrow P^{op}T^2 b \underset{P^{op}P\chi}{\overset{\epsilon_{P^{op}Tb}}{\underset{P^{op}P\epsilon_{P^{op}b}}{\longrightarrow}}} P^{op}Tb \underset{\chi}{\overset{\epsilon_{P^{op}b}}{\rightrightarrows}} P^{op}b$$

where $\chi = P^{op}f$, mapped to:

$$\cdots \longrightarrow T^3 b \underset{T^2 f}{\overset{P\epsilon_{P^{op}Tb}}{\underset{TP\epsilon_{P^{op}b}}{\longrightarrow}}} T^2 b \underset{Tf}{\overset{P\epsilon_{P^{op}b}(=\mu b)}{\rightrightarrows}} Tb \overset{f}{\dashrightarrow} b$$

an algebra structure on b

Partial (fat) geometric realizations

The fat geometric realization of a semi-simplicial diagram can be computed as the colimit of the sequence

$$I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \ldots$$

where I_n is the colimit of the restriction of the diagram to $i \leq n$.

In the case of an algebraic simplicial diagram, this sequence satisfies a relation that allows an inductive computation.

Partial semi-simplex categories

$$\Delta^{op}_{+,\leq 2} : \quad \dots \longrightarrow 2 \;\overset{d^2_0,\, d^2_1,\, d^2_2}{\Longrightarrow}\; 1 \;\overset{d^1_0,\, d^1_1}{\rightrightarrows}\; 0$$

$$\partial^{+}\Delta^{op}_{+,\leq 2} : \quad \dots \longrightarrow 2 \;\overset{d^2_0,\, d^2_1,\, d^2_2}{\Longrightarrow}\; 1 \;\overset{d^1_0,\, d^1_1}{\rightrightarrows}\; 0$$

$$\partial^{-}\Delta^{op}_{+,\leq 2} : \quad \dots \longrightarrow 2 \;\overset{d^2_0,\, d^2_1,\, d^2_2}{\Longrightarrow}\; 1 \;\overset{d^1_0,\, d^1_1}{\rightrightarrows}\; 0$$

$$\partial\Delta^{op}_{\leq 2} : \quad \dots \longrightarrow 2 \;\overset{d^2_0,\, d^2_1,\, d^2_2}{\Longrightarrow}\; 1 \;\overset{d^1_0,\, d^1_1}{\rightrightarrows}\; 0$$

Recursive relation for truncated realizations

$$
\begin{array}{ccc}
I_n & \longrightarrow & \partial^- I_n \times_{\partial I_n} \partial^+ I_n \\
\downarrow & & \downarrow \\
X_1 & \xrightarrow[\Delta]{} & X_1 \times_{\partial I_n} X_1
\end{array}
\qquad\qquad
\begin{array}{ccc}
\partial^- I_{n+1} & \longrightarrow & X_0 \\
\downarrow & & \downarrow \\
T I_n & \longrightarrow & T X_0
\end{array}
$$

$$\partial^+ I_n + 1 = \partial^+ I_n (= X_0) \qquad\qquad \partial I_{n+1} = T \partial^+ I_n \times X_0$$

Internal towers and their internal limit

Definition

A internal sequence is just a morphism $\mathbb{N} \to \mathcal{U}$:

$$
\begin{array}{ccccc}
u_k & \longrightarrow & \sum_{n \in \mathbb{N}} u_n & \longrightarrow & \tilde{\mathcal{U}} \\
\downarrow & & \downarrow & \lrcorner & \downarrow \\
* & \xrightarrow[s^k \circ 0]{} & \mathbb{N} & \xrightarrow[u]{} & \mathcal{U}
\end{array}
$$

Internal towers and their internal limit

Definition

An internal tower is a sequence equipped with a map f making the following diagram commute:

$$\begin{array}{ccc} * \amalg \sum_{n\in\mathbb{N}} u_n & \xleftarrow{\;f\;} & \sum_{n\in\mathbb{N}} u_n \\ \Big\downarrow{\scriptstyle id_* \amalg p} & & \Big\downarrow{\scriptstyle p} \\ * \amalg \mathbb{N} & \xrightarrow[<0,s>]{} & \mathbb{N} \end{array}$$

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The internal limit of an internal tower is the following equalizer:

$$u_\infty \dashrightarrow \sum_{n \in \mathbb{N}} u_n \underset{\iota_1}{\overset{f}{\rightrightarrows}} * \amalg \sum_{n \in \mathbb{N}} u_n$$

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Proposition

The internal limit of the internal tower of partial (co)geometric realizations for an algebraic diagram defines an (external) (co)geometric realization.

Wrapping up

The higher Beck (co)monadicity theorem can be applied so that $\mathcal{E}_\alpha^{\rightarrow}$ is equivalent to the category of coalgebras for the comonad at work.
It is remarkable that finite cocompleteness of $\mathbf{Poly}(\mathcal{E}_\alpha)$ can be deduced from only the lccc structure ($\mathbf{Poly}(\mathcal{E}_\alpha)$ embeds fully in $Hom_{Cat_\infty}(\mathcal{E}_\alpha^{op}, \mathcal{E}_\alpha^{op})$).

$\mathcal{E}$ can be recovered from the $(\mathcal{E}_\alpha)$ in a way that ensures it is finitely cocomplete too.

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Theorem (Elementary ∞-topos)

An (∞-)category $\mathcal{E}$ is an elementary ∞-topos iff it

- *is finitely complete*
- *is locally cartesian closed*
- *admits a subobject classifier*
- *admits enough "universes" (a.k.a object classifiers)*
- *admits a natural number object*

Thank you for your attention!

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