Čech Cohomology in Homotopy Type Theory

Ingo Blechschmidt, Felix Cherubini, David Wärn

Part of the synthetic algebraic geometry project:

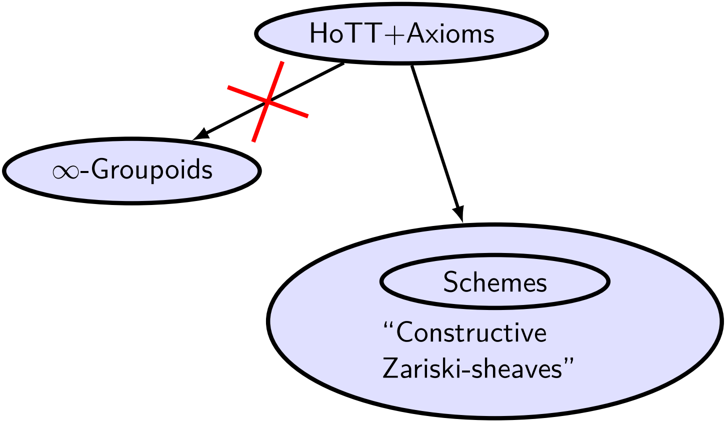
Synthetic Algebraic Geometry in the Zariski-Topos

Stay updated on synthetic algebraic geometry with the mailing list.

This is a latex documentation of our understanding of the synthetic /internal theory of the Zariski-Topos. There are currently the following parts:

- Foundations (draft pdf)
- Čech-Cohomology (early draft pdf)
- Differential Geometry/étale maps (early draft pdf)
- Proper Schemes (early draft pdf)
- Topology of Synthetic Schemes (early draft pdf)
- $\mathbb{A}^1$-homotopy theory (early draft pdf)
- Random Facts, i.e. a collection of everything that still needs to find a good place (almost empty pdf)

There is a related formalization project.



* Schemes = quasi-compact, quasi-separated schemes of finite type

Reminder: The 3 Axioms

Axiom: We have a local, commutative ring R.

For a finitely presented R-algebra A, define:

$$\mathrm{Spec}(A) :\equiv \mathrm{Hom}_{R\text{-algebra}}(A, R)$$

Axiom (synthetic quasi-coherence (SQC)):
For any finitely presented R-algebra A, the map

$$a \mapsto (\varphi \mapsto \varphi(a)) : A \xrightarrow{\sim} R^{\mathrm{Spec}(A)}$$

is an equivalence.
Axiom (Zariski-local choice):
For every surjective π, there merely exist local sections s_i

$$\begin{tikzcd} & E \arrow[d, two heads, "\pi"] \\ D(f_i) \arrow[r, hook] \arrow[ur, dashed, bend left, "s_i"] & \mathrm{Spec}(A) \end{tikzcd}$$

with $f_1, \dots, f_n : A$ coprime.

For $A : X \to \mathrm{Ab}$, define *cohomology* as:

$$H^n(X, A) :\equiv \left\| \prod_{x:X} K(A_x, n) \right\|_{\mathrm{set}}$$

For $A : X \to \mathrm{Ab}$, define *cohomology* as:

$$H^n(X, A) :\equiv \left\| \prod_{x:X} K(A_x, n) \right\|_{\mathrm{set}}$$

Good because:

- $\prod$-type.
- Homotopy group: $H^n(X, A) = \pi_k(\prod_{x:X} K(A, n+k))$.
- $\|\_\|_{\mathrm{set}}$ is a modality.

For $A : X \to \mathrm{Ab}$, define *cohomology* as:

$$H^n(X, A) :\equiv \left\| \prod_{x:X} K(A_x, n) \right\|_{\mathrm{set}}$$

Good because:

- $\prod$-type.
- Homotopy group: $H^n(X, A) = \pi_k(\prod_{x:X} K(A, n+k))$.
- $\|\_\|_{\mathrm{set}}$ is a modality.

Non-trivial for $X : \mathrm{Set}$ because:
$X =$ Pushout of sets $U \leftarrow Y \to V$,
Then a "cohomology class" $X \to K(A, 1)$ is given by:

- Maps $f : U \to K(A, 1)$, $g : V \to K(A, 1)$.
- And $h : (x : Y) \to f(x) = g(x)$, which is essentially a map $Y \to A$, if U and V don't have higher cohomology...

Mayer-Vietoris Sequence

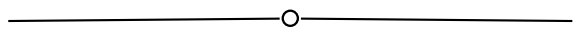
$X = U \cup V$ (or some other pushout). Then the following is exact:

$$
\begin{tikzcd}
0 \arrow[r] & H^0(X) \arrow[r] & H^0(U) \times H^0(V) \arrow[r] & H^0(U \cap V) \arrow[dll] \\
& H^1(X) \arrow[r] & H^1(U) \times H^1(V) \arrow[r] & H^1(U \cap V) \arrow[dll] \\
& H^2(X) \arrow[r] & H^2(U) \times H^2(V) \arrow[r] & \cdots
\end{tikzcd}
$$

Examples: Cohomology of Pushouts

$$\begin{tikzcd} \mathbb{A}^{\times} \arrow[r] \arrow[d] & \mathbb{A}^1 \arrow[d] \\ \mathbb{A}^1 \arrow[r] & \mathbb{A}^1_{\cdot\cdot} \end{tikzcd}$$

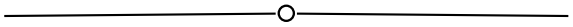
$\mathbb{A}^1_{\cdot\cdot}$



Examples: Cohomology of Pushouts

$$\begin{tikzcd} \mathbb{A}^\times \arrow[r] \arrow[d] & \mathbb{A}^1 \arrow[d] \\ \mathbb{A}^1 \arrow[r] & \mathbb{A}^1_{..} \end{tikzcd}$$

$\mathbb{A}^1_{..}$

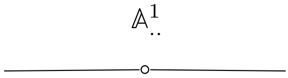


$$R[X] \times R[X] \longrightarrow R[X][X^{-1}] \longrightarrow\!\!\!\!\rightarrow H^1(\mathbb{A}^1_{..}, R) \qquad \neq 0$$

$$(P, Q) \mapsto P - Q$$

Examples: Cohomology of Pushouts

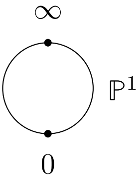
$$\begin{tikzcd}
\mathbb{A}^\times \arrow[r] \arrow[d] & \mathbb{A}^1 \arrow[d] \\
\mathbb{A}^1 \arrow[r] & \mathbb{A}^1_{\cdot\cdot}
\end{tikzcd}$$



$$R[X] \times R[X] \longrightarrow R[X][X^{-1}] \twoheadrightarrow H^1(\mathbb{A}^1_{\cdot\cdot}, R) \qquad \neq 0$$

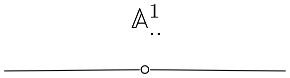
$$(P, Q) \mapsto P - Q$$

$$\begin{tikzcd}
\mathbb{A}^\times \arrow[r] \arrow[d, "\frac{1}{x}"'] & \mathbb{A}^1 \arrow[d] \\
\mathbb{A}^1 \arrow[r] & \mathbb{P}^1
\end{tikzcd}$$



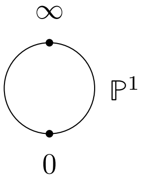
Examples: Cohomology of Pushouts

$$\begin{tikzcd} \mathbb{A}^{\times} \arrow[r] \arrow[d] & \mathbb{A}^1 \arrow[d] \\ \mathbb{A}^1 \arrow[r] & \mathbb{A}^1_{\cdot\cdot} \end{tikzcd}$$



$$R[X] \times R[X] \longrightarrow R[X][X^{-1}] \twoheadrightarrow H^1(\mathbb{A}^1_{\cdot\cdot}, R) \qquad \neq 0$$
$$(P, Q) \mapsto P - Q$$

$$\begin{tikzcd} \mathbb{A}^{\times} \arrow[r] \arrow[d, "\frac{1}{x}"'] & \mathbb{A}^1 \arrow[d] \\ \mathbb{A}^1 \arrow[r] & \mathbb{P}^1 \end{tikzcd}$$



$$R[X] \times R[X] \longrightarrow R[X][X^{-1}] \twoheadrightarrow H^1(\mathbb{P}^1, R) \qquad = 0$$
$$(P, Q) \mapsto P - Q(X^{-1})$$

Vanishing on Affine Schemes

Theorem

Let $X = \operatorname{Spec} A$, $n > 0$ *and* $M : X \to R\text{-}\mathrm{Mod}_{\mathrm{wqc}}$, *then*

$$H^n(X, M) = 0.$$

Vanishing on Affine Schemes

Theorem

Let $X = \operatorname{Spec} A$, $n > 0$ *and* $M : X \to R\text{-}\mathrm{Mod}_{\mathrm{wqc}}$, *then*

$$H^n(X, M) = 0.$$

Proof.

Uses: Homotopy theory, algebra, Zariski-local choice and properties of wqc-modules. $\square$

Applying Cohomology

To (merely) construct functions, one can use exactness of:

$$0 \to \prod_{x:X} M_x \to \prod_{x:X} N_x \to \prod_{x:X} K_x \to H^1(X, M)$$

(which holds if $0 \to M_x \to N_x \to K_x \to 0$ is exact for all $x : X$)

Applying Cohomology

To (merely) construct functions, one can use exactness of:

$$0 \to \prod_{x:X} M_x \to \prod_{x:X} N_x \to \prod_{x:X} K_x \to H^1(X, M)$$

(which holds if $0 \to M_x \to N_x \to K_x \to 0$ is exact for all $x : X$)

To show a scheme is affine!
This holds if for all $I : X \to R\text{-}\mathrm{Mod}_{\mathrm{wqc}}$ such that I_x is an ideal in R for all $x : X$, we have $H^1(X, I) = 0$, by a recent theorem of Ingo Blechschmidt and David Wärn.

Čech-Cohomology

Generalization of Mayer-Vietoris for spaces $X = \bigcup_{i=1}^{n} U_i$, where U_i and all their k-ary intersections have trivial higher cohomology.

Čech-Cohomology

Generalization of Mayer-Vietoris for spaces $X = \bigcup_{i=1}^n U_i$, where U_i and all their k-ary intersections have trivial higher cohomology.
By Vanishing, separated schemes with their open affine cover, are an example.

Čech-Cohomology

Generalization of Mayer-Vietoris for spaces $X = \bigcup_{i=1}^{n} U_i$, where U_i and all their k-ary intersections have trivial higher cohomology.
By Vanishing, separated schemes with their open affine cover, are an example.
Two proofs:

- By universal property, that both Čech and Eilenberg-MacLane cohomology have.
- Roughly, by pointwise turning the colimit $U_1(x) \vee \cdots \vee U_n(x)$ into a limit.

Thank you!