BIASED ELEMENTARY DOCTRINES AND QUOTIENT COMPLETIONS

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2nd International Conference on Homotopy Type Theory 2023
CMU, May 22

Outline

Elementary doctrines

$$\mathrm{P} : \mathscr{C}^{op} \to \mathrm{InfSL}$$

- $\mathscr{C}$ has strict finite products
- For every $X \in \mathscr{C}$ there exists an element $\delta_X \in P(X \times X)$ with

$$\mathrm{P}(Y \times X) \underset{\mathrm{P}_{\langle 1,2,2\rangle}}{\overset{\mathrm{P}_{<1,2>}(-)\wedge \mathrm{P}_{<2,3>}\delta_X}{\rightleftarrows}} \mathrm{P}(Y \times X \times X) \quad (\bot)$$

Equivalently:

1 $\top_X \leq \mathrm{P}_{\Delta_X}(\delta_X)$ $\quad\quad \vdash x = x$

2 $\mathrm{P}(X) = Des(\delta_X)$ $\quad\quad A(x_1), x_1 = x_2 \vdash A(x_2)$

3 $\delta_X \boxtimes \delta_Y \leq \delta_{X \times Y}$ $\quad\quad x_1 = x_2, y_1 = y_2 \vdash (x_1, y_1) = (x_2, y_2)$

Examples

- (*Variations*) If $\mathscr{C}$ has strict finite products and weak pullbacks then

$$\Psi_{\mathscr{C}} : \mathscr{C}^{op} \to \mathsf{InfSL}$$

$$\Psi_{\mathscr{C}}(X) := (\mathscr{C}/X)_{po}$$

$$\Psi_{\mathscr{C}}(f) := f^*$$

$$\delta_X = \lfloor \Delta_X \rceil$$

$$\begin{array}{ccc} P & \xrightarrow{f'} & M \\ {\scriptstyle f^*m}\downarrow & \lrcorner & \downarrow{\scriptstyle m} \\ Y & \xrightarrow[f]{} & X \end{array}$$

- (*Subobjects*) If $\mathscr{C}$ is a lex category

$$\mathrm{Sub}_{\mathscr{C}} : \mathscr{C}^{op} \to \mathsf{InfSL}$$

$$\mathrm{Sub}_{\mathscr{C}}(X) := \{\lfloor m \rceil \mid m : M \rightarrowtail X\}$$
$$\mathrm{Sub}_{\mathscr{C}}(f) := f^*$$
$$\delta_X = \lfloor \Delta_X \rceil$$

Examples from type theory I

- Let **ML** be the category of closed types and terms up to f.e. of intensional MLTT

$$F^{ML} : \mathbf{ML}^{op} \to \mathsf{InfSL}$$

$F^{ML}(X) := \{x : X \vdash B(x), \text{up to equiprovability}\}$
$F^{ML}(t)(B(x)) := B(t(y))$, for a term $y : Y \vdash t(y) : X$

- Let **mTT** be the intensional level of the Minimalist Foundation[1] and $\mathcal{CM}$ the syntactic category of *collections*

$$G^{\mathbf{mTT}} : \mathcal{CM}^{op} \to \mathsf{InfSL}$$

$G^{\mathbf{mTT}}(X) := \{B(x)\ prop\ (x \in X), w.r.t\ equiprovability\}$
$G^{\mathbf{mTT}}(t)(B(x)) := B(t(y))$, where $t(y) \in X\ (y : Y)$.

Examples from type theory II

- Let **hSets** be the category of h-sets arising from **HoTT**

$$H^{0,\text{-}1} : \mathbf{hSets}^{op} \to \mathrm{InfSL}$$

$$H^{0,\text{-}1}(X) = \{x : X \vdash B(x) |\ B(x) \text{ is an h-prop}\}$$

Rmk. $\delta_X = \mathrm{Id}_X$
Obs.: $\mathcal{CM}$ and **ML** have strict finite products and weak pullbacks.
Obs.: $F^{ML} \cong \Psi_{\mathbf{ML}}$.
Obs.: $H^{0,\text{-}1} \cong \mathrm{Sub}_{\mathbf{hSets}}$.

[1] [MS05] [Mai09]

Elementary quotient completion[2]

If $\mathrm{P} : \mathscr{C}^{op} \to \mathrm{InfSL}$ is an elementary doctrine:

- A P-eq. relation on $X \in \mathscr{C}$ is an element $\rho \in \mathrm{P}(X \times X)$ + ref.+ sym.+ trans.
- A quotient of ρ is an arrow $q : X \to C$ s.t. $\rho(x_1, x_2) \vdash q(x_1) = q(x_2)$ + universal property

$$\overline{\mathrm{P}} : \overline{\mathscr{C}}^{op} \to \mathrm{InfSL}$$

	$\overline{\mathscr{C}}$	$\overline{\mathrm{P}}$
Obj.	(X, ρ)	$\overline{\mathrm{P}}(X, \rho) := Des(\rho)^{\star}$
Arr.	$\lfloor f \rceil : (X, \rho) \to (Y, \sigma)$	$\overline{\mathrm{P}}\lfloor f \rceil := \mathrm{P}_f$

$${}^{\star}Des(\rho) = \{A(x) \in \mathrm{P}(X) | \rho(x_1, x_2), A(x_1) \vdash A(x_2)\}$$

[2] [MR13]

Examples

1. If $\mathscr{C}$ has strict finite products and weak pullbacks then:

$$\Psi_{\mathscr{C}} : \mathscr{C}^{op} \to \mathsf{InfSL} \qquad \overline{\Psi_{\mathscr{C}}} \cong \mathsf{Sub}_{\mathscr{C}_{ex/wlex}} : \mathscr{C}^{op}_{ex/wlex} \to \mathsf{InfSL}$$

Pseudo eq. relations $R \overset{r_1}{\underset{r_2}{\rightrightarrows}} X$ $\longleftrightarrow$ $\Psi_{\mathscr{C}}$-eq. relations $\lfloor < r_1, r_2 >: R \to X \times X \rceil$

2. $\overline{G^{\mathbf{mTT}}} : \overline{\mathcal{CM}}^{op} \to \mathsf{InfSL}$ provides the main example of e.q.c. that is not an exact completion. $\overline{G^{\mathbf{mTT}}}$ describes the interpretation of (extensional level) **emTT** into **mTT**.

3. $F^{ML} : \mathbf{ML}^{op} \to \mathsf{InfSL} \qquad \overline{F^{ML}} : \overline{\mathbf{ML}}^{op} \to \mathsf{InfSL} \qquad (\overline{\mathbf{ML}} \cong \mathbf{Std})$
4. $H^{0,\text{-}1} : \mathbf{hSets}^{op} \to \mathsf{InfSL}$ has quotients (due to quotient type).

My problem

Q. How can I deal with "dependent setoids" within this framework?

$$x : X \vdash B(x) \qquad \text{with} \qquad y, y' : B(x) \vdash y \sim_{B(x)} y'$$

I. Consider an arrow $\pi : S \to X$ in $\mathscr{C}$ and an element $\sim_B \in P(W)$

$$\begin{array}{ccc} W & \longrightarrow & S \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \pi} \\ S & \underset{\pi}{\longrightarrow} & X \end{array}$$

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P. We may have just *weak* pull-backs!

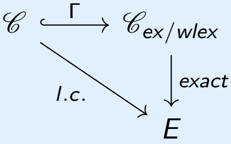
$$\begin{array}{ccc}
\sum_{s,s':S} \pi(s) =_X \pi(s') & \longrightarrow & S \\
\downarrow & & \downarrow \pi \\
S & \xrightarrow[\pi]{} & X
\end{array}$$

Rmk. $(B(x), \sim_{B(x)})$ must be *proof-irrelevant*
if $p, p' : x =_X x'$ then

$$\mathrm{tr}(p, b(x)) \sim_{B(x')} b(x') \qquad \text{iff} \qquad \mathrm{tr}(p', b(x)) \sim_{B(x')} b(x')$$

The categorical gap

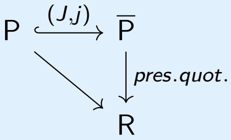
Thm. (Carboni-Vitale '98)
If $\mathscr{C}$ weakly lex the pre-composition with Γ



gives an equivalence

$$\mathbf{Lco}(\mathscr{C}, E) \cong \mathbf{EX}(\mathscr{C}_{ex/wlex}, E)$$

Thm. (Maietti-Rosolini '13)
If $\mathrm{P} : \mathscr{C}^{op} \to \mathrm{InfSL}$ is elementary the pre-composition with (J, j)



gives a natural equivalence

$$\mathbf{EqD}(\mathrm{P}, R) \cong \mathbf{QED}(\overline{\mathrm{P}}, R)$$

$$\mathscr{C} \xrightarrow[\text{strict finite products}]{\text{Only in case of}} \Psi_{\mathscr{C}} : \mathscr{C}^{op} \to \mathrm{InfSL}$$

Outline

My solution: Biased elementary doctrines

From now on $\mathscr{C}$ has weak finite products.

Definition

A functor $\mathrm{P} : \mathscr{C}^{op} \to \mathrm{InfSL}$ *is a biased elementary doctrine if for every* $X \in \mathscr{C}$ *and for every weak product* $X \xleftarrow{\mathrm{p}_1} W \xrightarrow{\mathrm{p}_2} X$ *there exists an element* $\delta^{(\mathrm{p}_1,\mathrm{p}_2)} \in \mathrm{P}(W)$ *satisfying:*

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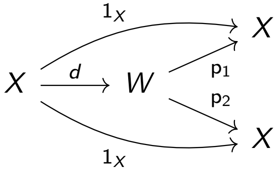
1) *For every commutative diagram*



we have

$\top_X \leq \mathrm{P}_d \delta^{(\mathrm{p}_1,\mathrm{p}_2)}$.

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2) $\mathrm{P}(X) = Des(\delta^{(\mathsf{p}_1,\mathsf{p}_2)})$, *i.e. for every* $\alpha \in \mathrm{P}(X)$

$$\mathrm{P}_{\mathsf{p}_1}\alpha \wedge \delta^{(\mathsf{p}_1,\mathsf{p}_2)} \leq \mathrm{P}_{\mathsf{p}_2}\alpha.$$

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3) *For any weak product* $X' \xleftarrow{\mathrm{p}'_1} W' \xrightarrow{\mathrm{p}'_2} X'$ *and for every commutative diagram*

$$\begin{array}{ccccc} & & X' & \xrightarrow{f} & X \\ & \overset{\mathrm{p}'_1}{\nearrow} & & & \overset{\mathrm{p}_1}{\nearrow} \\ W' & & \xrightarrow{g} & & W \\ & \underset{\mathrm{p}'_2}{\searrow} & & & \underset{\mathrm{p}_2}{\searrow} \\ & & X' & \xrightarrow[f]{} & X \end{array}$$

we have $\delta^{(\mathrm{p}'_1,\mathrm{p}'_2)} \leq \mathrm{P}_g \delta^{(\mathrm{p}_1,\mathrm{p}_2)}$.

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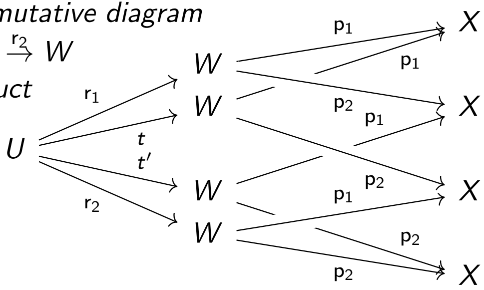
Definition

A functor $\mathrm{P} : \mathscr{C}^{op} \to \mathrm{InfSL}$ *is a biased elementary doctrine if for every* $X \in \mathscr{C}$ *and for every weak product* $X \xleftarrow{\mathrm{p}_1} W \xrightarrow{\mathrm{p}_2} X$ *there exists an element* $\delta^{(\mathrm{p}_1,\mathrm{p}_2)} \in \mathrm{P}(W)$ *satisfying:*

4) *For every commutative diagram where* $W \xleftarrow{\mathrm{r}_1} U \xrightarrow{\mathrm{r}_2} W$ *is a weak product*



we have $\delta^{(\mathrm{p}_1,\mathrm{p}_2)} \in Des(\mathrm{P}_t\delta^{(\mathrm{p}_1,\mathrm{p}_2)} \wedge \mathrm{P}_{t'}\delta^{(\mathrm{p}_1,\mathrm{p}_2)})$*, i.e.*

$$\mathrm{P}_{\mathrm{r}_1}\delta^{(\mathrm{p}_1,\mathrm{p}_2)} \wedge \mathrm{P}_t\delta^{(\mathrm{p}_1,\mathrm{p}_2)} \wedge \mathrm{P}_{t'}\delta^{(\mathrm{p}_1,\mathrm{p}_2)} \leq \mathrm{P}_{\mathrm{r}_2}\delta^{(\mathrm{p}_1,\mathrm{p}_2)}.$$

Examples I

1. Every elementary doctrine P is a biased elementary doctrine. If $X \overset{p_1}{\leftarrow} W \overset{p_2}{\rightarrow} X$ is a weak product then there exists a unique arrow $< p_1, p_2 >: W \to X \times X$

$$\delta^{(p_1,p_2)} := P_{<p_1,p_2>}\delta_X$$

2. If $\mathscr{C}$ is wlex then the functor $\Psi_{\mathscr{C}} : \mathscr{C}^{op} \to \text{InfSL}$ is a biased elementary doctrine and

$$\delta^{(p_1,p_2)} := \lfloor e \rceil$$

where

$$E \xrightarrow{e} W \underset{p_2}{\overset{p_1}{\rightrightarrows}} X$$

is a weak equalizer of p_1, p_2.

Examples II

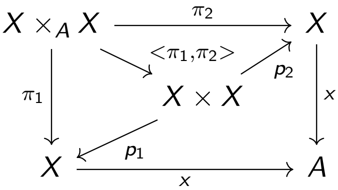
3. If $\mathrm{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is a (biased) elementary doctrine with weak comprehensions and comprehensive diagonals and $A \in \mathscr{C}$ then the *slice doctrine* is a biased elementary doctrine:

$$\mathrm{P}_{/A} : \mathscr{C}/A^{op} \to \mathsf{InfSL}$$

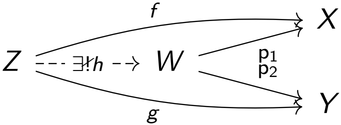
$\mathrm{P}_{/A}(x : X \to A) := \mathrm{P}(X)$
$\mathrm{P}_{/A}(f : y \to x) := \mathrm{P}_f$

$\mathrm{P}_{/A}(w) = \mathrm{P}(X \times_A X)$
where $w := x\pi_1 = y\pi_2$ and

$$\delta^{(\pi_1,\pi_2)} := \mathrm{P}_{<\pi_1,\pi_2>}\delta_X$$



Key differences with "strict" elementary doctrines

- Two weak products $X \xleftarrow{\mathrm{p}_1} W \xrightarrow{\mathrm{p}_2} Y$ and $X \xleftarrow{\mathrm{p}'_1} W' \xrightarrow{\mathrm{p}'_2} Y$ are not necessarily isomorphic.
- The fibers $\mathrm{P}(W)$ and $\mathrm{P}(W')$ are not necessarily isomorphic.
- 

  The reindexings P_h and $\mathrm{P}_{h'}$ are not necessarily equal.
- We have only the inequality

$$\delta_{X\times Y} \leq \delta_X \boxtimes \delta_Y$$

Intuition: $x_1 = x_2, y_1 = y_2 \not\Rightarrow ((x_1, y_1), p) = ((x_2, y_2), q)$

$\delta_{X\times Y} \sim$ *proof-relevant* equality
$\delta_X \boxtimes \delta_Y \sim$ *proof-irrelevant* or *component-wise* equality

Proof-irrelevant elements

Definition

$X \overset{p_1}{\leftarrow} W \overset{p_2}{\rightarrow} Y$ weak product. The *proof-irrelevant (p.i.)* elements of W are the sub-poset of $\mathsf{P}(W)$ given by $\text{P-Irr}(W) := Des(\delta_X \boxtimes \delta_Y)$[3]

- Different weak products (of X, Y) have isomorphic proof-irrelevant elements: take an arrow $W' \xrightarrow{h} W$ s.t. $p_i \circ h = p'_i$

$$\begin{array}{ccc} \text{P-Irr}(W) & \xrightarrow{\cong} & \text{P-Irr}(W') \\ \downarrow & & \downarrow \\ \mathsf{P}(W) & \xrightarrow[\mathsf{P}_h]{} & \mathsf{P}(W') \end{array}$$

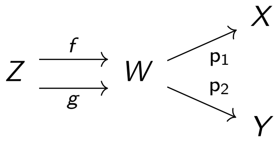
- Up to iso: we denote proof-irrelevant elements of X and Y with $\mathsf{P}^s[X, Y]$.
- Proof-irrelevant elements are *reindexed by projections*.

[3] Some work to prove that the definition depends only on W.

Elements reindexed by projections

Definition

Let $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ *a functor from* $\mathscr{C}$ *with weak finite products and let* $X \overset{\mathsf{p}_1}{\leftarrow} W \overset{\mathsf{p}_2}{\to} X$ *be a weak product. An element* $\alpha \in \mathsf{P}(W)$ *is reindexed by projections (r.b.p.) if for every commutative diagram*



$$P_f(\alpha) = P_g(\alpha).$$

Obs.: $\delta^{(\mathsf{p}_1, \mathsf{p}_2)}$ is r.b.p. !

Obs.: In case of full weak comprehensions r.b.p. elements coincide with *p.i.* elements.

Main examples

- In $F^{ML}_{/A} : \mathbf{ML}/A^{op} \to \mathsf{InfSL}$, if

$$\begin{array}{ccc} W := \sum_{x:X, y:Y} \mathsf{Id}_A(f(x), g(y)) & \xrightarrow{\pi_2} & Y \\ {\scriptstyle \pi_1}\downarrow & & \downarrow {\scriptstyle g} \\ X & \xrightarrow[f]{} & A \end{array}$$

$$F^{ML}_{/A}\text{-}\mathsf{Irr}(W) = \{(x, y, p) : W \vdash R(x, y, p) |$$

$$\mathsf{Id}_X(x, x'),\ \mathsf{Id}_Y(y, y'),\ R(x, y, p)\ \vdash R(x', y', p')\}.$$

- If $\mathscr{C}$ is wlex and $X \overset{p_1}{\leftarrow} W \overset{p_2}{\rightarrow} Y$ is a weak product, the proof-irrelevant elements of $\Psi_{\mathscr{C}}(W)$ are:

Theorem

If $\mathscr{C}$ is weakly left exact, then

$$\Psi_{\mathscr{C}}\text{-}\mathsf{Irr}(W) \cong (\mathscr{C}/(X, Y))_{po}.$$

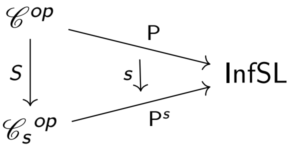
Strictification...

If $\mathscr{C}$ is a category, we can freely[4] add strict finite products and obtain the category $\mathscr{C}_s$:

Obj. are finite lists $[X_i]_{i\in[n]}$

Arr. $(f,\hat{f}) : [X_i]_{i\in[n]} \to [Y_j]_{j\in[m]}$

If $\mathrm{P} : \mathscr{C}^{op} \to \mathrm{InfSL}$ is a b.e.d. then we can build P^s using p.i. elements



Theorem

If P is a b. e. d. then $\mathrm{P}^s \in \mathbf{ED}$. Vice versa, if $R : \mathscr{C}_s^{op} \to \mathrm{InfSL}$ in **ED**, the pre-composition $R \circ S : \mathscr{C}^{op} \to \mathrm{InfSL}$ is a b. e. d.

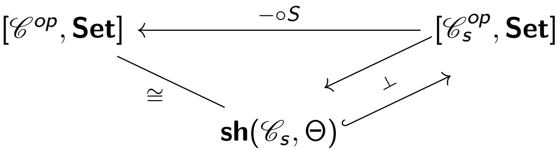
[4] Obs: Weak products are neither preserved nor "strictified" by S.

...and sheafification[5]

If $\mathscr{C}$ has weak finite products then consider the Grothendieck topology Θ on $\mathscr{C}_s$ generated by singleton families

$$\{\mathsf{p} : [W] \to [X_1, \ldots, X_n]\}$$

where $\mathsf{p}_i : W \to X_i$, for $1 \leq i \leq n$, weak products of $X_1, \ldots, X_n \in \mathscr{C}$



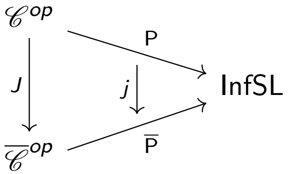
[5] *Grothendieck topologies and weak limits for constructive mathematics.* Ongoing work with: J. Emmenegger, F. Pasquali and G. Rosolini.

Extending elementary quotient completion

If $\mathrm{P} : \mathscr{C}^{op} \to \mathrm{InfSL}$ is a b.e.d. a P-eq. relation[6] over $X \in \mathscr{C}$ is a $\rho \in \mathrm{P}^s[X, X]$ satisfying ref., sym. and tra.. The category $\overline{\mathscr{C}}$:

Obj. Pairs (X, ρ)

Arr. $\lfloor f \rceil : (X, \rho) \to (Y, \sigma)$ are $f : X \to Y$ s.t. $\rho \le \mathrm{P}^s_{[f]\times[f]}(\sigma)$.



Theorems

1) $\overline{\mathrm{P}} \in \mathbf{QED}$
2) $\circ(J, j) : \mathbf{QED}(\overline{\mathrm{P}}, R) \cong \mathbf{Lco}(\mathrm{P}, R)$, for every $R \in \mathbf{QED}$

Obs: $\overline{\mathrm{P}} \not\cong \overline{\mathrm{P}^s}$.

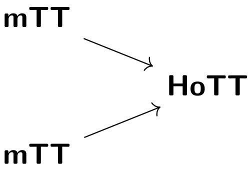
[6] The usual notion relies on strict fine products!

Applications

- (Elimination of the problem) Thm. $\overline{\mathsf{P}_{/A}} \cong \overline{\mathsf{P}}_{/(A,\delta_{[A]})}$.
- (Filling the gap) $\mathscr{C}_{ex/wlex}$ and the e.q.c. are instances of this construction since $\Psi_{\mathscr{C}}$-eq. relation coincides with *per* (cones + ref. + sym. + trans.)
- We can define $\implies$, $\exists$ and $\forall$-biased elementary doctrines.
- Full generalization of the result of Carboni, Rosolini and Emmenegger about the lcc of the *ex/wlex* exact completion.

Current research / Future work

- Extending elementary quotient completion to a richer framework such as *cwf, comprehension categories....*
- Use the results obtained to describe categorically the compatibility[7] of the Minimalist foundation with HoTT



[7] [CM23]

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