A Type Theoretic Model of Synthetic Algebraic Geometry

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This talk

Part 1: definition of schemes

Part 2: model of a dependent type version of Zariski topos

We formulate an axiomatisation of the Zariski ∞-topos

In this setting, we define what is a (quasi-compact quasi-separated) scheme

Part 1: Definition of schemes

Lawvere's message on the Category mailing list, March 2003

The 1973 Buffalo Colloquium talk by Alexander Grothendieck had as its main theme that the 1960 definition of scheme (which had required as a prerequisite the baggage of prime ideals and the spectral space, sheaves of local rings, coverings and patchings, etc.), should be abandoned as *the* fundamental *one and replaced by the simple idea of a good functor from rings to sets. The needed restrictions could be more intuitively and more geometrically stated directly in terms of the topos of such functors, and of course the ingredients from the "baggage" could be extracted when needed as auxiliary explanations of already existing objects, rather than being carried always as core elements of the very definition.*

Definition of schemes

Thus his definition is essentially well-known, and indeed is mentioned in such texts as Demazure-Gabriel, Waterhouse, and Eisenbud; but it is not carried through to the end, resulting in more complication, rather than less.

To recall some well-known ingredients, let A be the category of finitely-presented commutative algebras over k... Then the underlying set functor R on A serves as the "line", and any system of polynomial equations with coefficients in k determines also a functor (subspace of R^n) in the well-known way; in fact, the idea of spec is simply identified with the Yoneda embedding of A^{op}.

The problem

Fix a ring k

Consider for base category the opposite of the category of f.p. k-algebra

Consider the Zariski topology on this category

Grothendieck's suggestion is to define a scheme as a "good" sheaf

All the problem is to have a good description of "good"

As Lawvere writes, all definitions in the literature are somewhat complex

Definition of schemes as functor of points

Literature: books by Demazure-Gabriel, Eisenbud-Harris, Jantzen

Blog post of David Madore

None of these works use internal language, in contrast to

Anders Kock *Linear algebra and projective geometry in the Zariski topos* Aarhus Preprint, 74/75

Ph.D. thesis of Ingo Blechschmidt: *Using the internal language of toposes in algebraic geometry*, 2017

Internal language

We present a definition expressed in the internal language of the Zariski topos

Furthermore, it is done axiomatically from 3 axioms

And this is expressed in the language of *dependent type theory*

Internal language

The Zariski topos is the classifying topos of the theory of local ring

We have the generic ring R, represented by $k[X]$

This ring is (internally) a local ring

Anders Kock noticed that it satisfies (unexpected) non geometric properties

Unexpected since they are not provable from the theory of local rings

E.g. if $r \neq 0$ in R then r is invertible!

More generally if $\neg(r_1 = \cdots = r_n = 0)$ then one r_i is invertible

Some notations

$Sp(A)$ for f.p. R-algebra A and $Sp(A) = Hom(A, R)$

We use letters $A, B, \ldots$ for finitely presented R-algebras

$D(r)$ is the proposition that r is invertible

Define $Cov(A)$ the type of unimodular vectors $f_1, \ldots, f_n$

R local means that any unimodular vector has an invertible component

Note that $Sp(R_r) = D(r)$ and $Sp(R/(r))$ is the proposition $r = 0$

Internal Language

Anders Kock *Linear algebra and projective geometry in the Zariski topos* Aarhus Preprint, 74/75

The definition of projective space $\mathbb{P}^n$ is natural *internally*: set of lines in R^{n+1}

I think that the constructed objects are the same as those "classically" considered (in Demazure-Gabriel, say, [2] p. 9)

If we look at it *externally*, we get that $\mathbb{P}^n(A)$ is the set of submodule of A^{n+1} that are factor direct and locally free of rank 1

Much more complex external description

The 3 axioms

If f in A, define $D(f) = \Sigma_{x:Sp(A)} D(xf)$

$D(f)$, with the injection π_1, defines a subset of $Sp(A)$

It is natural to write $f(x)$ instead of xf!!

If P family of types over $Sp(A)$ and $c = f_1, \ldots, f_n$ in $Cov(A)$ define $\Pi_c P$ to be the product of $\Pi_{(x,\_):D(f_i)} P(x)$

The 3 axioms

Here are the 3 axioms for SAG (Synthetic Algebraic Geometry)

A f.p. R-algebra

The 3 axioms

Axiom 1: R is a local ring

Axiom 2: the canonical map $A \to R^{Sp(A)}$ is an isomorphism

Axiom 3: $Sp(A)$ satisfies "Zariski local choice"

$$(\Pi_{x:Sp(A)} \|P\|) \to \left\|\Sigma_{c:Cov(A)}\Pi_c P\right\|$$

for *any* family of types P over $Sp(A)$

The 3 axioms

These 3 axioms seem to be "complete"

All desired properties for developing the (basic) theory of schemes follow from these 3 axioms

We do need the framework of dependent type theory with univalence and propositional truncation

For instance, for studying cohomology we need to use Axiom 3 with P of arbitrary homotopy level (Eilenberg-MacLane space)

Definition of schemes

We define *internally* when a proposition ψ is open

$\exists_{r_1,\ldots,r_n:R}\ \psi \leftrightarrow D(r_1) \vee \cdots \vee D(r_n)$

In dependent type theory

$\|\Sigma_{r_1,\ldots,r_n:R}\ \psi \leftrightarrow D(r_1) \vee \cdots \vee D(r_n)\|$

Definition of schemes

We can prove from the 3 axioms

Theorem: *If* $\psi(x)$ $(x : Sp(A))$ *is a family of open propositions then there exists* $f_1, \dots, f_m$ in A *such that*

$$\psi(x) \leftrightarrow D(f_1 x) \vee \cdots \vee D(f_m x)$$

This expresses a strong *uniformity* condition

Definition of schemes

A priori we only have

$\forall_x \exists_{r_1,\dots,r_n}\ \psi(x) \leftrightarrow D(r_1) \vee \cdots \vee D(r_n)$

and the theorem states that we can replace this by

$\exists_{f_1,\dots,f_n} \forall_x\ \psi(x) \leftrightarrow D(f_1 x) \vee \cdots \vee D(f_n x)$

Such a family $\psi(x)$, which is *pointwise* open defines an *open subset* of $Sp(A)$

Definition of schemes

An *affine scheme* is a type X such that there exists A satisfying $X = Sp(A)$

Then X is a set and A is uniquely determined!

So, in dependent type theory, this can be expressed with Σ

This follows from Axiom 2: $Sp(A)$ determines A

Definition of schemes

Definition: X *is a* scheme *iff it is a finite union of open subsets that are affine schemes*

This implies that X is a set

The main point is to have a suitable definition of open subsets

And this we have with the classifier of open propositions

A scheme is a type satisfying a *property*

Examples

$\mathbb{A}^n = Sp(R[X_1, \ldots, X_n])$ is an affine scheme

We define $\mathbb{P}^n$ as the set of lines in R^{n+1}

This is a scheme, union of $n+1$ affine schemes $\mathbb{A}^n$

If X and Y are schemes then so is $X \times Y$

We get exactly the *quasi-compact quasi-separated* schemes

The intersection of two affine open is a finite union of affine open

Dependent Type Theory

We want to express these axioms in *dependent* type theory

In this setting, we have access to Eilenberg-MacLane spaces (e.g. using David Wärn's elegant recursive definition)

We can define cohomology groups $H^n(X, R) = \pi_0(K(R, n)^X)$

Cf. Urs Schreiber mathoverflow 2009 answer

I like to say that there is only a single abstract definition of coomology: in any $(\infty, 1)$-topos given objects X and A the cohomology of X with coefficients in A is the connected components of the hom-∞-groupoid A^X

So we need a model of univalent type theory which is a model of the 3 axioms

Dependent Type Theory

In particular, in SAG one can show that $H^n(X, R) = 0$ for $n > 0$ if X affine scheme (this application suggested the axiom of Zariski local choice)

We have $H^n(X, M) = 0$ if X affine and M is weakly quasi-coherent R-module, i.e. such that the canonical map $M_r \to M^{D(r)}$ is an isomorphism

(And Ingo and David have even an internal proof that if X is a scheme and $H^1(X, M) = 0$ for all such M then X is affine, which is an internal version of Serre's cohomological criterion for affine schemes)

And we can compute $H^1(X, M)$ via Čech if X is a scheme

Some history

Grothendieck's letter to Larry Breen, 17.2.1975

The construction of the cohomology of a topos in terms of integration of stacks makes no appeal at all to complexes of abelian sheaves and still less to the technique of injective resolutions. One has the impression that in this spirit, via the definition (which remains to be made explicit!) of n-stacks, it is all related above all to the "Čechist" calculations in terms of hypercoverings. Now these last are written with the help of a small dose of semi-simplicial algebra.

Čech cohomology and Eilenberg-MacLane

Felix will explain how to connect the definition via Eilenberg-MacLane spaces and the Čech definition for schemes

Some history

I do not know if a theory of stacks and of operations on them can be written without ever using semi-simplicial algebra. If yes, there would be essentially three distinct approaches for constructing the cohomology of a topos

a) viewpoint of complexes of sheaves, injective resolutions, derived categories (commutative homological algebra)

b) viewpoint Čechist or semi-simplicial (homotopical algebra)

c) viewpoint of n-stacks (categorical algebra, or non-commutative homological algebra).

In (a) one "resolves" the coefficients, in (b) one resolves the base space (or topos), and in (c) it appears one resolves neither the one nor the other.

Some history

Grothendieck

Sur quelques points d'algèbre homologique, Tôhoku Math. J. (2) 9 (1957), p. 119-221.

Joyal's letter 11.4.1984 presents a model structure for simplicial sheaves

Voici donc une démonstration basée sur des principes qui ne sont pas éloignés de ceux que tu as utilisés dans ton article "Tôhoku"

"Principe de localisation booléenne"

use of Barr's Theorem

Comment

Barr's Theorem is needed since one uses classical logic to build the usual model structure for simplicial sets

Here we make use that we have a constructive/essentially algebraic way to build a model of dependent type theory with univalence

This "relativizes" directly to presheaf models

We can then use the theory of left exact modalities to build a sheaf model

Cf. *Higher modalities*, Shulman 2012

Part 2: Constructive models of univalence

Parametrised by two presheaves Φ and $\mathbf{I}$

Φ classifies cofibration

$\mathbf{I}$ is the interval

Using Φ we can define contractibility

We also can define propositional truncation

New kind of inductive data types, constructors satisfying restriction equations

Constructive models of univalence

No problem to get *presheaf* models since these models are done in a constructive metatheory

This is to be compared to the situation with simplicial sets

In particular we can consider the presheaf model over the (opposite of the) category of f.p. k-algebras

Presheaf model

We get a model of another theory with 3 axioms SAG'

Axiom 1': R is a ring

Axiom 2': The canonical map $A \to R^{Sp(A)}$ is an isomorphism

Axiom 3': $Sp(A)$ satisfies choice

The last axiom means $(\Pi_{x:Sp(A)} \|P(x)\|) \to \left\|\Pi_{x:Sp(A)} P(x)\right\|$

And this for *any* family of types $P(x)$

Sheaf model

We explain now how to build *purely internally* a sheaf model satisfying the theory SAG from *any* model satisfying the theory SAG'

The 3 axioms

SAG (Synthetic Algebraic Geometry)

Axiom 1: R is a local ring

Axiom 2: The canonical map $A \to R^{Sp(A)}$ is an isomorphism

Axiom 3: $Sp(A)$ satisfies "Zariski local choice"

$$(\Pi_{x:Sp(A)} \|P\|) \to \left\|\Sigma_{c:Cov(A)} \Pi_c P\right\|$$

for *any* family of types P over $Sp(A)$

Presheaf model

SAG'

Axiom 1': R is a ring

Axiom 2': The canonical map $A \to R^{Sp(A)}$ is an isomorphism

Axiom 3': $Sp(A)$ satisfies choice

Sheaf model/Forcing

We “force” R to be a local ring

$Cov(R)$ unimodular sequences $r_1, \ldots, r_n$

Consider the family of propositions

$\|D(r_1) + \cdots + D(r_n)\|$

for $(r_1, \ldots, r_n)$ in $Cov(R)$

We force all these propositions to be contractible

Sheaf model/Forcing

Model of "modal" types, i.e. types T such that the diagonal map

$T \to T^{\|D(r_1)+\cdots+D(r_n)\|}$ is an equivalence

Sheaf model

In a purely internal way, the modal types form a model of type theory with univalence and propositional truncation (and more generally HITs)

Sheaf model

The fact that R is modal follows from the fact that the canonical map

$$R_r \to R^{D(r)}$$

is an isomorphism

This follows from Axiom 2' with $A = R_r$, since $D(r) = Sp(R_r)$

In this new model $\perp$ is the modal proposition $0 =_R 1$ in the presheaf model

So this model is non trivial if k is non trivial

Ongoing work

All this is part of ongoing work with Peter Arndt, Ingo Blechschmidt, Hugo Moeneclaey, Josselin Poiret, and David Wärn.

Foundations	(Felix, Matthias, Thierry)
Proper Schemes	(David, Felix, Matthias, Thierry)
Differential Geometry	(David, Felix, Hugo, Matthias)
Čech Cohomology	(David, Felix, Ingo)
Formalization	(Felix, Josselin, Matthias)
$\mathbb{A}^1$-*homotopy theory*	(Peter, David, Felix, Hugo)