Directed Types are Weak ω-Categories

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Homotopy Type Theory 2023, May 25

Outline

1. Types are weak ω-groupoids
2. Globular multicategories with homomorphism types
3. Directed type are weak ω-categories

Types are Weak ω-Groupoids

Types are Weak ω-Groupoids

When we have identity types, each type comes equipped with the *algebraic structure of an ω-groupoid*.

$$\begin{aligned}
&\vdash \mathsf{refl}_a : \mathsf{Id}_A(a,a) \\
p : \mathsf{Id}_A(a,b), q : \mathsf{Id}_A(b,c) &\vdash p \circ_0 q : \mathsf{Id}_A(a,c)
\end{aligned}$$

$$\begin{aligned}
p : \mathsf{Id}_A(a,b) &\vdash \mathsf{refl}_p : \mathsf{Id}_{\mathsf{Id}_A(a,b)}(p,p) \\
\phi : \mathsf{Id}_{\mathsf{Id}_A(a,b)}(p,q), r : \mathsf{Id}_A(b,c) &\vdash \phi \circ_0 r : \mathsf{Id}_{\mathsf{Id}_A(a,c)}(p \circ_0 r, q \circ_0 r) \\
\phi : \mathsf{Id}_{\mathsf{Id}_A(a,b)}(p,q), \psi : \mathsf{Id}_{\mathsf{Id}_A(a,b)}(q,s) &\vdash \phi \circ_1 \psi : \mathsf{Id}_{\mathsf{Id}_A(a,b)}(p,s) \\
&\vdots
\end{aligned}$$

Globular Higher Categories

An ω-category is:

- A globular collection of objects, arrows, arrows between arrows, . . . , together with coherent composition operations

[1] Michael Batanin. "Monoidal Globular Categories As a Natural Environment for the Theory of Weak n-Categories". In: *Advances in Mathematics* 136.1 (1998), pp. 39 –103.

Globular Higher Categories

An ω-category is:

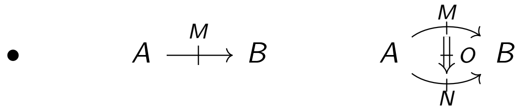
- A globular collection of objects, arrows, arrows between arrows, . . . , together with coherent composition operations
- More precisely, an algebra of a contractible globular operad[1]

[1] Michael Batanin. "Monoidal Globular Categories As a Natural Environment for the Theory of Weak n-Categories". In: *Advances in Mathematics* 136.1 (1998), pp. 39 –103.

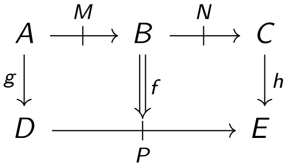
Globular Multicategories

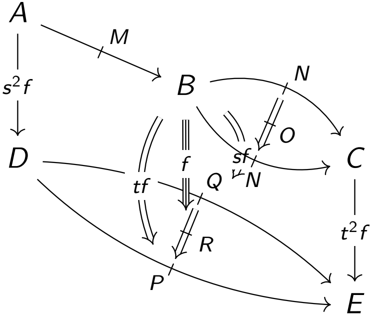
A *globular multicategory* consists of:

- A globular set of labelled globes called *n-types*

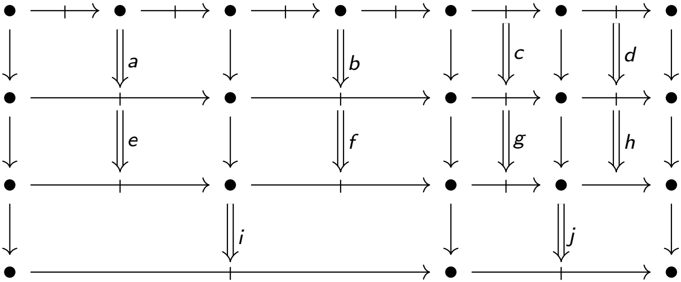


- A globular set of *terms*, abstract assignments sending pasting diagrams of types to types





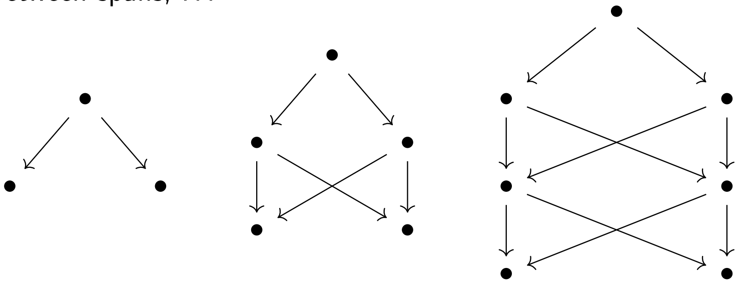
- Terms can be composed, and this composition is associative and unital.



Spans

Let $\mathcal{C}$ be a category with pullbacks. There is a globular multicategory $\mathrm{Span}\,\mathcal{C}$ such that

- 0-types are objects in $\mathcal{C}$, 1-types are spans, 2-types are spans between spans, ...

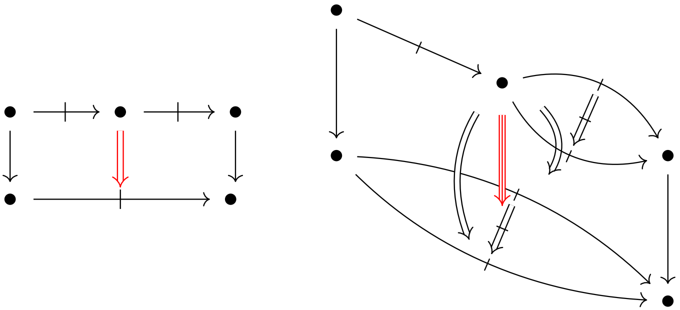


- Terms are natural transformations, where we first paste together source types using pullbacks.

Globular Operads

Definition

A *globular operad* is a globular multicategory with a unique n-type for each n. A globular operad is contractible if each term boundary has a filler.



Globular ω-Categories

Definition

An ω*-category* in a globular multicategory $\mathbb{X}$ consists of a normalised contractible globular operad $\mathbb{T}$ together with a homomorphism

$$\mathbb{T} \longrightarrow \mathbb{X}.$$

Identity Type Categories

An *identity type category* is a category $\mathcal{C}$ with classes $\mathcal{I}, \mathcal{F} \subseteq \text{Arr } C$, of *acyclic cofibrations* and *fibrations*, satisfying:

- **Fibrancy:** $\mathcal{C}$ has a terminal object $\top$, the canonical morphisms to $\top$ are fibrations.
- **Composition:** $\mathcal{I}$ and $\mathcal{F}$ are closed under composition and contain identities.
- **Stability:** The pullback of a fibration along an arbitrary morphism in $\mathcal{C}$ exists, and is a fibration
- **Frobenius:** The pullback of an acyclic cofibration along a fibration is an acyclic cofibration.
- **Orthogonality:** Acyclic cofibrations have the left-lifting property with respect to fibrations.
- **Identity Types:** For each fibration $f : M \twoheadrightarrow A$, the diagonal map $\Delta_f : M \to M \times_A M$ factorises into a composite

$$M \xhookrightarrow{r_M} \text{Id}_M \xrightarrow{g} M \times_A M$$

Types are Weak ω-Categories

Proof[2][3].

Let $(\mathcal{C}, \mathcal{I}, \mathcal{F})$ be a an identity type category. Then there is a globular multicategory $\mathsf{Span}(\mathcal{C}, \mathcal{F})$ such that a type in $\mathsf{Span}(\mathcal{C}, \mathcal{F})$ is a higher span whose legs are in $\mathcal{F}$.

For each object X in $\mathcal{C}$ (that is, each type in our type theory), the collection of types and terms built from X using the introduction and elimination rules for identity types define a normalised contractible globular operad.

$$\mathbb{T}_X \hookrightarrow \mathsf{Span}(\mathcal{C}, \mathcal{F})$$

□

[2] Benno van den Berg and Richard Garner. "Types are weak ω-groupoids". In: *Proceedings of the London Mathematical Society* 102.2 (2011), pp. 370–394.

[3] Peter LeFanu Lumsdaine. "Weak omega-categories from intensional type theory". In: *Logical Methods in Computer Science* Volume 6, Issue 3 (2010).

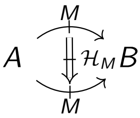
What is it about $\mathrm{Span}(\mathcal{C}, \mathcal{F})$ that allows this proof to work?

Globular Multicategories with Homomorphism Types

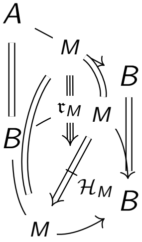
Definition

A globular multicategory has *homomorphism types* when each n-type $A \xrightarrow{M} B$ is equipped with:

- an $(n+1)$-type



- an $(n+1)$-term



- We have analogues of the introduction and elimination rules for identity types that let us add/remove homomorphism types to/from pasting diagrams.

Globular Multicategories with Homomorphism Types

Theorem

Let $\mathbb{T}$ be the free globular multicategory with homomorphism types generated by a 0-type. Then $\mathbb{T}$ is a normalised contractible globular operad.

Corollary

Each 0-type A in a globular multicategory with homomorphism types $\mathbb{X}$ has the structure of a weak ω-category.

Proof.

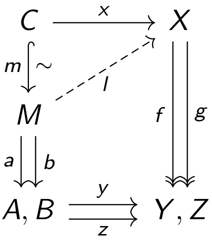
By adjointness A induces a canonical homomorphism

$$\mathbb{T} \longrightarrow \mathbb{X}.$$

□

Span($\mathcal{C}, \mathcal{F}$) has homomorphism types

- The homomorphism type introduction and elimination rules in the globular multicategory Span($\mathcal{C}, \mathcal{F}$) require certain fillers:

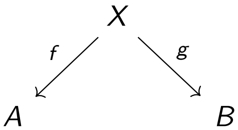


The axioms of identity type categories imply that the marked arrows are acyclic cofibrations and fibrations. The fillers exist by orthogonality of $\mathcal{I}$ and $\mathcal{F}$.

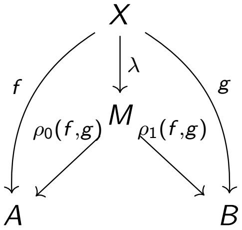
Two-sided Factorisation Systems

Definition (North)

A *two-sided factorisation* of a span



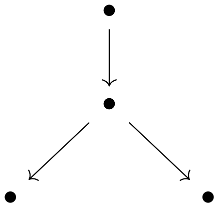
is a commutative diagram of the following form:



Two-sided Factorisation Systems

Definition (North)

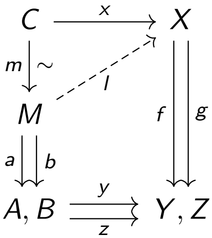
A *sprout* is a diagram of the following form:



Two-sided Factorisation Systems

Definition (North)

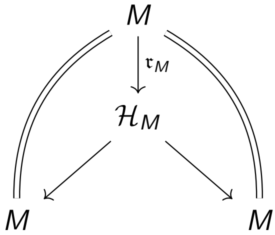
A sprout *lifts* against a span if any diagram of the following form has a filler:



Homomorphism Type Categories

A *homomorphism type category* is a category C, a class $\mathcal{F}$ of spans called *two-sided fibrations*, a class $\mathcal{R}$ of sprouts called *representors*, satisfying:

- **Identities:** Identity sprouts are representors.
- **Composition:** Two-sided fibrations and representors are closed under composition.
- **Lifting:** Representors lift against two-sided fibrations
- **Homomorphism Types:** For each two-sided fibration $A \overset{M}{\nrightarrow} B$, the trivial span on M factorises into a representor



in $\mathcal{C}/_{A\times B}$ such that $\mathcal{H}_M$ is a two-sided fibration.

Homomorphism Type Categories

Example

Every identity type category is a homomorphism type category.

Example

The category of categories is a homomorphism type category*.

- Two-sided fibrations are two-sided fibrations.
- Representors are those sprouts which lift against two-sided fibrations.
- Homomorphism types are directed path categories .

*We need an extra condition that requires morphisms between fibrations to be cartesian, or we need to restrict to discrete fibrations.

Directed Types are Weak ω-Categories

Theorem

Every homomorphism type category induces a globular multicategory with homomorphism types.

Directed Types are Weak ω-Categories

Theorem

Every homomorphism type category induces a globular multicategory with homomorphism types.

Corollary

Each object in a homomorphism type category has the structure of a weak ω-category.

Bonus

- For any globular multicategory $\mathbb{X}$, we can freely add homomorphism types.
- We can also prove analogous theorems such as "Terms are weak ω-functors" and dependent types are ""weak ω-profunctors".

Thank you.