Relating ordinals in set theory to ordinals in type theory

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Overview

Working inside HoTT, we have two goals:

1. Show that the set-theoretic ordinals coincide with the type-theoretic ordinals.
   - By set-theoretic ordinal we mean a hereditarily transitive set as in constructive set theory (Powell'75, Aczel–Rathjen'10).
   - By type-theoretic ordinal we mean the ordinals developed in the HoTT Book and further by Escardó and collaborators in the Agda development TypeTopology.
2. Generalize the above correspondence to *all* sets in the cumulative hierarchy by considering certain extensional wellfounded relations.

Ordinals in homotopy type theory

- In HoTT, a (type-theoretic) ordinal is defined as a type X with a prop-valued binary relation $<$ that is transitive, extensional and wellfounded.

- Extensionality means that we have

$$x = y \iff \forall(u : X).(u < x \iff u < y)$$

  It follows that X is an hset.

- Wellfoundedness is defined in terms of accessibility, but is equivalent to the assertion that for every $P : X \to \mathcal{U}$, we have $\Pi(x : X).P(x)$ as soon as $\Pi(x : X).(\Pi(y : X).(y < x \to P(y))) \to P(x)$.

- For example, $(\mathbb{N}, <)$ is a type-theoretic ordinal.

The ordinal of type-theoretic ordinals

- We write Ord for the type of (small) type-theoretic ordinals.
- We make this type into a (large) type-theoretic ordinal itself: The relation $\prec$ on Ord given by

$$\begin{aligned} \alpha \prec \beta &\iff \alpha \text{ is an initial segment of } \beta \\ &\iff \Sigma(y : \beta).(\alpha = \beta \downarrow y) \end{aligned}$$

  is transitive, wellfounded and extensional, where we write $\beta \downarrow y$ for the (sub)ordinal $\Sigma(x : \beta).(x < y)$.

Ordinals in set theory

- Def. A set x is transitive if for every $y \in x$ and $z \in y$, we have $z \in x$.
- Def. A set-theoretic ordinal is a transitive set whose elements are all transitive.
- Lemma The elements of a set-theoretic ordinal are again set-theoretic ordinals.
  Thus, a set is a set-theoretic ordinal if and only if it is hereditarily transitive.
- Ex. The sets $\emptyset$, $\{\emptyset\}$ and $\{\emptyset, \{\emptyset\}\}$ are all set-theoretic ordinals, but $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$ isn't, as $\{\{\emptyset\}\}$ is a non-transitive member.

The cumulative hierarchy in HoTT

- HoTT hosts a model $(\mathbb{V}, \in)$ of a constructive set theory, known as the cumulative hierarchy.

  The type $\mathbb{V}$ is a HIT with point-constructor

  $$\mathbb{V}\text{-set}(A, f) : \mathbb{V} \quad \text{for} \quad A : \mathcal{U} \text{ and } f : A \to \mathbb{V}$$

  quotiented by bisimilarity: $\mathbb{V}\text{-set}(A, f)$ and $\mathbb{V}\text{-set}(B, g)$ are identified exactly when f and g have the same image.

- For example, the empty set is represented by $\mathbb{V}\text{-set}(\mathbf{0}, \mathbf{0}\text{-rec})$, and if $x : \mathbb{V}$, then the singleton $\{x\}$ is represented by $\mathbb{V}\text{-set}(\mathbf{1}, \lambda(u : \mathbf{1}).x)$.

The ordinal of set-theoretic ordinals

- We define set-membership $\in : \mathbb{V} \to \mathbb{V} \to \mathsf{Prop}$ by

$$x \in \mathbb{V}\text{-set}(A, f) \;:\equiv\; \exists(a : A).f(a) = x$$

- Using $\in$, we define the subtype $\mathbb{V}_{\mathsf{ord}}$ of $\mathbb{V}$ of set-theoretic ordinals in HoTT.

- The cumulative hierarchy $\mathbb{V}$ validates the axioms of $\in$-extensionality and $\in$-induction.

  Since $\mathbb{V}_{\mathsf{ord}}$ is restricted to hereditarily transitive sets, we get:

  $(\mathbb{V}_{\mathsf{ord}}, \in)$ is a type-theoretic ordinal.

Set-theoretic and type-theoretic ordinals coincide

- Thm. The type-theoretic ordinals $(\mathbb{V}_{\mathsf{ord}}, \in)$ and $(\mathsf{Ord}, \prec)$ are equal.

Thus, in HoTT,
set-theoretic and type-theoretic ordinals coincide.

From type-theoretic ordinals to set-theoretic ordinals

- Define $\Phi : \mathsf{Ord} \to \mathbb{V}_{\mathsf{ord}}$ by transfinite recursion:

$$\Phi(\alpha) :\equiv \mathbb{V}\text{-set}(\alpha, \lambda(a : \alpha).\Phi(\alpha \downarrow a)).$$

- This is well-defined, because $(\alpha \downarrow a) \prec \alpha$ (by definition of $\prec$) and the fact that $\prec$ is wellfounded.

From set-theoretic ordinals to type-theoretic ordinals

- The map $\Psi : \mathbb{V}_{\mathrm{ord}} \to \mathrm{Ord}$ is the rank function:

$$\Psi(\mathbb{V}\text{-set}(A, f)) :\equiv \bigvee_{a:A} (\Psi(f(a)) + \mathbf{1}),$$

  where $\bigvee$ denotes the supremum of ordinals, which may be constructed as a quotient of the sum $\Sigma_{a:A}(\Psi(f(a)) + 1)$.

- It is possible to give nonrecursive descriptions of the rank:

$$\Psi(x) \simeq \Sigma(y : \mathbb{V}).y \in x \quad \text{and} \quad \Psi(\mathbb{V}\text{-set}(A, f)) = A/{\sim},$$

  where $a \sim b \iff f(a) = f(b)$. (But be careful about size.)

Set-theoretic and type-theoretic ordinals coincide

- Thm. The type-theoretic ordinals $(\mathbb{V}_{\mathsf{ord}}, \in)$ and $(\mathsf{Ord}, \prec)$ are equal.
- Proof sketch The maps $\Phi : \mathsf{Ord} \to \mathbb{V}_{\mathsf{ord}}$ and $\Psi : \mathbb{V}_{\mathsf{ord}} \to \mathsf{Ord}$ give an isomorphism of ordinals. In particular,

$$\alpha \prec \beta \iff \Phi(\alpha) \in \Phi(\beta) \quad \text{and} \quad x \in y \iff \Psi(x) \prec \Psi(y).$$

Capturing all of the cumulative hierarchy

- Can we realize the *full* cumulative hierarchy $\mathbb{V}$ as a type of ordered structures?
  That is, can we find a type making the square

$$\begin{array}{ccc} \mathbb{V}_{\mathsf{ord}} & \xrightarrow{\simeq} & \mathsf{Ord} \\ \big\downarrow & & \big\downarrow \\ \mathbb{V} & \xrightarrow{\simeq} & ? \end{array}$$

  commute?

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- An initial naive attempt may be to simply drop transitivity, i.e., to take

  $$? = \text{type of extensional wellfounded relations.}$$

Why extensionality and wellfoundedness are not enough

- The two elements $\emptyset$ and $\{\emptyset\}$ are present in both the sets $\{\emptyset, \{\emptyset\}\}$ and $\{\{\emptyset\}\}$.
- But there is only *one* two-element extensional and wellfounded relation, namely $0 < 1$.
- Therefore, we consider extensional wellfounded relations $(A, <)$ with a marking: a predicate on A that picks out the top-level elements of a set.
- For example, for $\{\emptyset, \{\emptyset\}\}$ we mark *both* 0 and 1, but for $\{\{\emptyset\}\}$ we *only* mark 1.
- A marking is covering if any element can be reached from a marked element, i.e., if the relation contains no "junk".

Covered marked extensional wellfounded relations

- We write $\mathsf{MEWO}_{\mathsf{cov}}$ for the type of covered marked extensional wellfounded order relations.
- Every ordinal can be equipped with the trivial covering by marking everything. Thus, the type Ord of ordinals is a subtype of $\mathsf{MEWO}_{\mathsf{cov}}$.
- The idea of encoding sets as wellfounded structures isn't new, cf. Osius'74, Aczel'77 and '88, Taylor'96, Adamek et al.'13.
- The above approach worked well for our purposes of generalizing the theory of ordinals.

Capturing the full cumulative hierarchy

- The pair $(\mathbb{V}, \in)$ is a (trivially covered) mewo, thanks to $\in$-extensionality and $\in$-induction.

- The type $\mathsf{MEWO}_{\mathsf{cov}}$ is a (large, trivially covered) mewo itself: The relation $\prec$ on $\mathsf{MEWO}_{\mathsf{cov}}$ given by

$$A \prec B \iff \Sigma(y : B_{\mathsf{marked}}).(A = B \downarrow^{+} y)$$

  is wellfounded and extensional, where we write $B \downarrow^{+} y$ for the mewo $\Sigma(x : B).(x <^{+} y)$ whose marked elements are precisely the immediate predecessors of y.

- We get the bottom isomorphism by generalizing the constructions used to establish $\mathbb{V}_{\mathsf{ord}} \simeq \mathsf{Ord}$:

$$\begin{array}{ccc} \mathbb{V}_{\mathsf{ord}} & \xrightarrow{\simeq} & \mathsf{Ord} \\ \big\downarrow & & \big\downarrow \\ \mathbb{V} & \xrightarrow{\simeq} & \mathsf{MEWO}_{\mathsf{cov}} \end{array}$$

From mewos to $\mathbb{V}$-sets

- Recall the map $\Phi : \mathsf{Ord} \to \mathbb{V}_{\mathsf{ord}}$ defined as

$$\Phi(\alpha) :\equiv \mathbb{V}\text{-set}\,(\alpha, \lambda(a : \alpha).\Phi(\alpha \downarrow a)).$$

- Similarly, we define $\tilde{\Phi} : \mathsf{MEWO}_{\mathsf{cov}} \to \mathbb{V}$ as

$$\tilde{\Phi}(A) :\equiv \mathbb{V}\text{-set}\left(A, \lambda(a : A).\tilde{\Phi}(A \downarrow^{+} a)\right).$$

- The diagram

$$\begin{array}{ccc} \mathsf{Ord} & \xrightarrow{\ \Phi\ } & \mathbb{V}_{\mathsf{ord}} \\ \big\downarrow & & \big\downarrow \\ \mathsf{MEWO}_{\mathsf{cov}} & \xrightarrow{\ \tilde{\Phi}\ } & \mathbb{V} \end{array}$$

  commutes.

From $\mathbb{V}$-sets to mewos

- Recall the map $\Psi : \mathbb{V}_{\mathsf{ord}} \to \mathsf{Ord}$ defined as

$$\Psi(\mathbb{V}\text{-set}(A, f)) :\equiv \bigvee_{a:A} (\Psi(f(a)) + \mathbf{1}),$$

where $\bigvee$ denotes the supremum of ordinals.

- To emulate the above for $\mathbb{V}$ and $\mathsf{MEWO}_{\mathsf{cov}}$, we introduce unions and singletons of mewos.

- We then define $\tilde{\Psi} : \mathbb{V} \to \mathsf{MEWO}_{\mathsf{cov}}$ as

$$\tilde{\Psi}(\mathbb{V}\text{-set}(A, f)) :\equiv \bigcup_{a:A} (\{\tilde{\Psi}(f(a))\}).$$

Singleton mewos

- Translated to set-theory, the successor operation $(-)+1$ on ordinals corresponds to $S \mapsto S \cup \{S\}$.

  The union is necessary to ensure transitivity.

- Given a mewo X, we define the singleton mewo $\{X\}$:
  Its carrier is $X + \mathbf{1}$, its relation is given by
  - $\mathsf{inl}\, x < \mathsf{inl}\, y \iff x < y$,
  - $\mathsf{inl}\, x < \mathsf{inr}\, \star \iff x$ is marked,
  - $\mathsf{inr}\, \star < y$ is false for all y, and

  with $\mathsf{inr}\, \star$ the only marked element.

- <u>Lemma</u> If X is covered, then so is $\{X\}$.

The full cumulative hierarchy and covered mewos coincide

- <u>Thm</u>. The covered mewos $(\mathsf{MEWO}_{\mathsf{cov}}, \prec)$ and $(\mathbb{V}, \in)$ are equal.

- The theorem *generalizes* the correspondence between ordinals, as witnessed by commutative diagram

$$\begin{array}{ccc} (\mathbb{V}_{\mathsf{ord}}, \in) & \xrightarrow{\simeq} & (\mathsf{Ord}, \prec) \\ \big\downarrow & & \big\downarrow \\ (\mathbb{V}, \in) & \xrightarrow{\simeq} & (\mathsf{MEWO}_{\mathsf{cov}}, \prec) \end{array}$$

Conclusion

- In HoTT, the set-theoretic ordinals in $\mathbb{V}$ coincide with the type-theoretic ordinals.
- By generalizing from type-theoretic ordinals to covered mewos, we capture all sets in $\mathbb{V}$.
- Question: Do the type-theoretic ordinals in the cubical sets model of HoTT coincide with the set-theoretic ordinals?
- Question: Can we similarly capture non-wellfounded sets as certain graphs in HoTT?
- *Set-Theoretic and Type-Theoretic Ordinals Coincide*. TdJ, Nicolai Kraus, Fredrik Nordvall Forsberg and Chuangjie Xu. `arXiv:2301.10696`. Accepted for presentation at *LICS'23*. Fully formalized in AGDA.

References

[1] Peter Aczel. 'An introduction to inductive definitions'. In: *Handbook of Mathematical Logic*. Ed. by Jon Barwise. Vol. 90. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Company, 1977, pp. 739–782. DOI: 10.1016/S0049-237X(08)71120-0.

[2] Peter Aczel. *Non-well-founded sets*. CSLI lecture notes 14. Center for the Study of Language and Information, 1988.

[3] Peter Aczel and Michael Rathjen. 'Notes on Constructive Set Theory'. Book draft, available at: https://www1.maths.leeds.ac.uk/~rathjen/book.pdf. 2010.

[4] Jiří Adámek et al. 'Well-Pointed Coalgebras'. In: *Logical Methods in Computer Science* 9.3 (2013). DOI: 10.2168/LMCS-9(3:2)2013.

[5] Martín Hötzel Escardó et al. 'Ordinals in univalent type theory in Agda notation'. Agda development, HTML rendering available at: https://www.cs.bham.ac.uk/~mhe/TypeTopology/Ordinals.index.html. 2018.

[6] André Joyal and Ieke Moerdijk. *Algebraic Set Theory*. London Mathematical Society Lecture Note Series. Cambridge University Press, 1995. DOI: 10.1017/CBO9780511752483.

[7] William C. Powell. 'Extending Gödel's negative interpretation to ZF'. In: *The Journal of Symbolic Logic* 40.2 (1975), pp. 221–229. DOI: 10.1017/jsl.2017.84.

[8] Paul Taylor. 'Intuitionistic Sets and Ordinals'. In: *The Journal of Symbolic Logic* 61.3 (1996), pp. 705–744.