Duality for Clans and the Fat Small Object Argument

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Special session in honor of André Joyal's contributions to HoTT on his 80th birthday

Draft: https://github.com/jonas-frey/pdfs/blob/master/clan-duality.pdf

Overview

1. Finite-product theories and finite-limit theories
2. Clans
3. Models in Higher Types
4. The Fat Small Object Argument

Finite-product theories and finite-limit theories

Functorial semantics

Idea of functorial semantics:

- Theories are categories, models are functors!

More precisely:

- Logical theories $\mathbb{T}$ correspond to structured categories $\mathcal{C}[\mathbb{T}]$
- Models of $\mathbb{T}$ correspond to structure-preserving functors $\mathcal{C}[\mathbb{T}] \to \mathsf{Set}$
- Different kinds of theory correspond to different kinds of structure

Functorial semantics – algebraic theories

- For every **algebraic theory** $\mathbb{T}$ (like the theories of groups or rings) there's a finite-product category $\mathcal{C}[\mathbb{T}]$ (called **Lawvere theory**) such that

$$\mathbb{T}\text{-}\mathbf{Mod} \;\simeq\; \mathbf{FP}(\mathcal{C}[\mathbb{T}], \mathsf{Set}).$$

- $\mathcal{C}[\mathbb{T}]$ can be constructed 'out of syntax', and we have

$$\mathcal{C}[\mathbb{T}] \overset{\text{op}}{\simeq} \{\text{finitely generated free } \mathbb{T}\text{-models}\} \overset{\text{full}}{\subseteq} \mathbb{T}\text{-}\mathbf{Mod}.$$

Functorial semantics – essentially algebraic theories

- For every **essentially algebraic theory** $\mathbb{T}$ (like the **theory of categories**) there's a finite-limit category $\mathcal{L}[\mathbb{T}]$ such that
$$\mathbb{T}\text{-}\mathbf{Mod} \;\simeq\; \mathbf{FL}(\mathcal{L}[\mathbb{T}], \mathsf{Set}).$$

- Again, we can think of $\mathcal{L}[\mathbb{T}]$ as a 'syntactic category', and additionally we have
$$\mathcal{L}[\mathbb{T}] \overset{\mathsf{op}}{\simeq} \{\text{finitely presented } \mathbb{T}\text{-models}\} = \{\text{compact } \mathbb{T}\text{-models}\} \overset{\text{full}}{\subseteq} \mathbb{T}\text{-}\mathbf{Mod}$$
where a $A \in \mathbb{T}\text{-}\mathbf{Mod}$ is called **compact** if
$$\mathbb{T}\text{-}\mathbf{Mod}(A, -) : \mathbb{T}\text{-}\mathbf{Mod} \to \mathsf{Set}$$
preserves filtered colimits.

Duality for finite-limit theories

- The categories of models of essentially algebraic theories are precisely the **locally finitely presentable categories**[1], and we get a perfect correspondence between 'theories' and 'categories of models':

Theorem (Gabriel–Ulmer duality)

There's a **biequivalence of 2-categories**

$$\mathbf{FL} \underset{\{\text{compact objects}\}^{\mathrm{op}} \leftarrow\!\shortmid\, \mathcal{X}}{\overset{\mathcal{L} \mapsto \mathbf{FL}(\mathcal{L},\mathrm{Set})}{\longleftrightarrow}} \mathbf{LFP}^{\mathrm{op}}$$

between the 2-category **FL** of small finite-limit categories, and the 2-category **LFP** of locally finitely presentable categories.

- P. Gabriel and F. Ulmer. *Lokal präsentierbare Kategorien*. Springer-Verlag, 1971.

[1] i.e. locally small cocomplete categories with a dense set of compact objects

Duality for finite-product theories

An analogous duality for finite-product theories has only been formulated more recently, I found it in[2].

Theorem

There is a **biequivalence of 2-categories**

$$\mathbf{FP}_{cc} \underset{\{\text{compact projectives}\}^{op} \leftarrow\!\shortmid \mathcal{X}}{\overset{\mathcal{C} \mapsto \mathbf{FP}(\mathcal{C},\mathrm{Set})}{\longleftrightarrow}} \mathbf{ALG}^{op}$$

where

- $\mathbf{FP}_{cc}$ is the 2-category of small idempotent-complete finite-product categories
- **ALG** is the 2-category of **algebraic categories** and **algebraic functors**
  - An **algebraic category** is an l.f.p. category which is Barr-exact and where the compact (regular) projective objects are dense
  - An **algebraic functor** is a functor that preserves small limits, filtered colimits, and regular epimorphisms.

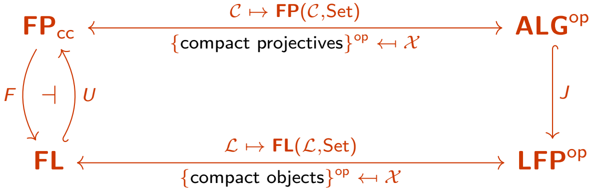
- We can recover finite-product theories only up to idempotent-completion, since we have to approximate 'free' by 'projective'.

[2] J. Adámek, J. Rosický, and E.M. Vitale. *Algebraic theories: a categorical introduction to general algebra.* Cambridge University Press, 2010.

Comparing the dualities

Finite-product duality is a special case of finite-limit duality, since

- finite-limit theories are more general than finite-product theories, and
- algebraic categories are locally finitely presentable.



Clan-duality can be viewed as a **refinement** of GU-duality which allows to control the amount of limit-preservation in the models.

Clans

Clans

Definition

A **clan** is a small category $\mathcal{T}$ with a terminal object 1, equipped with a class $\mathcal{T}^\dagger \subseteq \mathrm{mor}(\mathcal{T})$ of morphisms – called **display maps** and written $\twoheadrightarrow$ – such that

1. pullbacks of display maps along all maps exist and are display maps
$$\begin{array}{ccc} \Delta^+ & \xrightarrow{s^+} & \Gamma^+ \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ \Delta & \xrightarrow{s} & \Gamma \end{array},$$
2. display maps are closed under composition, and
3. isomorphisms and terminal projections $\Gamma \twoheadrightarrow 1$ are display maps.

- Observation: clans have finite products (as pullbacks over 1).
- Definition due to Taylor[3], name due to Joyal[4] (2017) ('a clan is a collection of families')
- Relation to semantics of dependent type theory: display maps represent **type families**.

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[3] P. Taylor. "Recursive domains, indexed category theory and polymorphism". PhD thesis. University of Cambridge, 1987, ğ 4.3.2.

[4] A. Joyal. "Notes on clans and tribes". In: *arXiv preprint arXiv:1710.10238* (2017).

Examples

- Finite-product categories $\mathcal{C}$ can be viewed as clans with $\mathcal{C}^\dagger = \{\text{product projections}\}$
- Finite-limit categories $\mathcal{L}$ can be viewed as clans with $\mathcal{L}^\dagger = \mathrm{mor}(\mathcal{L})$
- The syntactic category of every Cartmell-style **generalized algebraic theory** is a clan.
- For example, the **clan** $\mathcal{K}$ **for categories** is the syntactic category of the **GAT for categories**:
    - $\vdash\ O\quad \mathsf{type}$
    - $x\, y : O\ \vdash\ A(x,y)\quad \mathsf{type}$
    - $x : O\ \vdash\ \mathrm{id}(x) : A(x,x)$
    - $x\, y\, z : O\,,\ f : A(x,y)\,,\ g : A(y,z)\ \vdash\ g \circ f : A(x,z)$
    - $w\, x\, y\, z : O\,,\ e : A(w,x)\,,\ f : A(x,y)\,,\ g : A(y,z)\ \vdash\ (g \circ f) \circ e = g \circ (f \circ e) : A(w,z)$
    - $x\, y : O\,,\ f \in A(x,y)\ \vdash\ 1 \circ f = f = f \circ 1 : A(x,y)$

  Alternatively, $\mathcal{K}$ can be described semantically as dual to a category of finitely presented models:

$$\mathcal{K} = \{\text{categories free on finite graphs}\}^{\mathsf{op}} \subseteq \mathbf{Cat}^{\mathsf{op}}$$

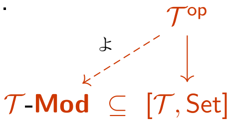
$$\mathcal{K}^\dagger = \{\text{functors induced by graph inclusions}\}^{\mathsf{op}}$$

Models

Definition

A **model** of a clan $\mathcal{T}$ is a functor $A : \mathcal{T} \to \mathsf{Set}$ which preserves 1 and pullbacks of display-maps.

- The category $\mathcal{T}$-**Mod** $\subseteq [\mathcal{T}, \mathsf{Set}]$ of models is l.f.p. and contains $\mathcal{T}^{\mathsf{op}}$.
- For FP-clans $(\mathcal{C}, \mathcal{C}^{\dagger})$ we have $(\mathcal{C}, \mathcal{C}^{\dagger})$-**Mod** $=$ **FP**$(\mathcal{C}, \mathsf{Set})$.
- For FL-clans $(\mathcal{L}, \mathcal{L}^{\dagger})$ we have $(\mathcal{L}, \mathcal{L}^{\dagger})$-**Mod** $=$ **FL**$(\mathcal{L}, \mathsf{Set})$.
- $(\mathcal{K}, \mathcal{K}^{\dagger})$-**Mod** $=$ **Cat**.



Observation

The same category of models may be represented by different clans.
For example, ordinary algebraic theories can be represented by FP-clans as well as FL-clans.

The weak factorization system

- Since distinct clans can have equivalent categories of models, $\mathcal{T}$ cannot be reconstructed from $\mathcal{T}$-**Mod** alone.
- Solution: equip $\mathcal{T}$-Mod additional structure in form of a **weak factorization system**.

Definition

Let $\mathcal{T}$ be a clan and よ $: \mathcal{T}^{\mathrm{op}} \to \mathcal{T}\text{-}\mathbf{Mod}$. Define w.f.s. $(\mathcal{E}, \mathcal{F})$ on $\mathcal{T}$-**Mod**:

$$\mathcal{F} = \mathbf{RLP}(よ(\mathcal{T}^{\dagger})) \qquad \text{'full maps'}$$

$$\mathcal{E} = \mathbf{LLP}(\mathcal{F}) \qquad \text{'extensions'}$$

Call $A \in \mathcal{T}\text{-}\mathbf{Mod}$ a 0-**extension**, if $(0 \to A) \in \mathcal{E}$.

- Hom-algebras $よ(\Gamma) = \mathcal{T}(\Gamma, -)$ are 0-extensions since all $\Gamma \twoheadrightarrow 1$ are display maps.
- The same weak factorization system was also introduced by S. Henry[5], see also[6].

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[5] S. Henry, *The language of a model category*, HoTTEST seminar, Jan. 2020, https://youtu.be/7_X0qbSXlfk

[6] S. Henry. "Algebraic models of homotopy types and the homotopy hypothesis". In: *arXiv preprint arXiv:1609.04622* (2016).

Full maps

- $f : A \to B$ in $\mathcal{T}$-**Mod** is full iff it has the RLP with respect to all $\text{よ}(p)$ for display maps $p : \Delta \twoheadrightarrow \Gamma$.

$$
\begin{array}{ccc}
\mathcal{T}(\Gamma,-) & \longrightarrow & A \\
{\scriptstyle \text{よ}(p)=\mathcal{T}(p,-)}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\
\mathcal{T}(\Delta,-) & \longrightarrow & B
\end{array}
\qquad\qquad
\begin{array}{ccc}
A(\Delta) & \xrightarrow{f_\Delta} & B(\Delta) \\
{\scriptstyle A(p)}\downarrow & & \downarrow{\scriptstyle B(p)} \\
A(\Gamma) & \xrightarrow{f_\Gamma} & B(\Gamma)
\end{array}
$$

- This is equivalent to display-naturality-squares being weak pullbacks.
- Considering $p : \Delta \twoheadrightarrow 1$ we see that full maps are surjective and hence regular epis.
- For FL-clans, only isos are full (consider naturality square for diagonal $\Delta \to \Delta \times \Delta$)
- For FP-clans we have

$$
\begin{array}{rcl}
\text{full map} & = & \text{regular epimorphism} \\
0\text{-extension} & = & \text{projective object}
\end{array}
$$

Duality for clans

Theorem

There is a bi-equivalence of 2-categories

$$\mathbf{Clan}_{\mathrm{cc}} \quad \underset{\mathcal{T} \mapsto \mathcal{T}\text{-}\mathbf{Mod}}{\overset{\mathfrak{C}(\mathfrak{X})^{\mathrm{op}} \leftarrow\!\shortmid \mathfrak{X}}{\longleftrightarrow}} \quad \mathbf{cAlg}^{\mathrm{op}}$$

where

- $\mathbf{Clan}_{\mathrm{cc}}$ is the 2-category of Cauchy complete clans,
- $\mathbf{cAlg}$ is the 2-category of **clan-algebraic categories**, i.e. l.f.p. categories $\mathfrak{X}$ equipped with an 'extension/full' WFS $(\mathcal{E}, \mathcal{F})$ such that
    1. the **full subcategory** $\mathsf{CZE}(\mathfrak{X}) \subseteq \mathfrak{X}$ **on compact** 0**-extensions** is dense in $\mathfrak{X}$,
    2. $(\mathcal{E}, \mathcal{F})$ is cofibrantly generated by maps in $\mathsf{CZE}(\mathfrak{X})$, and
    3. $\mathfrak{X}$ has **full and effective quotients of componentwise-full equivalence relations**.
- *Left to right*: $\mathcal{T}$-$\mathbf{Mod}$ is clan-algebraic for every clan $\mathcal{T}$,
- *Right to left*: for $\mathfrak{X}$ clan-algebraic, $\mathsf{CZE}(\mathfrak{X}) \subseteq \mathfrak{X}$ is a **coclan** with extensions as codisplay maps

Proof sketch

- For the proof we have to show that
  1. $\mathcal{T} \simeq \mathsf{CZE}(\mathcal{T}\text{-}\mathsf{Mod})^{\mathsf{op}}$ for all Cauchy-complete clans $\mathcal{T}$, and
  2. $\mathsf{CZE}(\mathfrak{X})^{\mathsf{op}}\text{-}\mathsf{Mod} \simeq \mathfrak{X}$ for all clan-algebraic categories $\mathfrak{X}$.
- For 2 we use a Reedy factorization on 2-truncated semi-simplicial algebras
- For 1 we use the **fat small object argument**, which implies that:

Lemma

$\mathsf{elts}(A)$ *is filtered for all* 0*-extensions* $A \in \mathcal{T}\text{-}\mathsf{Mod}$*, thus* 0*-extensions are* ***flat****.*

Models in Higher Types

Models in higher types

Let $\mathcal{S}$ be the ∞-topos of spaces/types.

Let $\mathcal{C}[\mathsf{Mon}]$ be the finite-product theory of monoids, and let $\mathcal{L}[\mathsf{Mon}]$ be the finite-limit theory of monoids. Then

$$\mathbf{FP}(\mathcal{C}[\mathsf{Mon}], \mathsf{Set}) \simeq \mathbf{FL}(\mathcal{L}[\mathsf{Mon}], \mathsf{Set}) \simeq \mathbf{Mon}$$

but $\mathbf{FP}(\mathcal{C}[\mathsf{Mon}], \mathcal{S})$ and $\mathbf{FL}(\mathcal{L}[\mathsf{Mon}], \mathcal{S})$ are different:

- $\mathbf{FL}(\mathcal{L}[\mathsf{Mon}], \mathcal{S})$ is just the category of monoids
- $\mathbf{FP}(\mathcal{C}[\mathsf{Mon}], \mathcal{S})$ is the ∞-category 'A_∞-algebras', i.e. homotopy-coherent monoids.

Moral

By being 'slimmer', finite-product theories leave room for higher coherences when interpreted in higher types.

This phenomenon was discussed under the name 'animation' in[7], and earlier in[8]

[7] K. Cesnavicius and P. Scholze. "Purity for flat cohomology". In: *arXiv preprint arXiv:1912.10932* (2019).
[8] D. Quillen. *Homotopical algebra*. Springer, 1967.

Four clan-algebraic weak factorization systems on Cat

Cat admits several clan-algebraic weak factorization systems:

- $(\mathcal{E}_1, \mathcal{F}_1)$ is cofib. generated by $\{(0 \to 1), (2 \to \mathbb{2})\}$
- $(\mathcal{E}_2, \mathcal{F}_2)$ is cofib. generated by $\{(0 \to 1), (2 \to \mathbb{2}), (2 \to 1)\}$
- $(\mathcal{E}_3, \mathcal{F}_3)$ is cofib. generated by $\{(0 \to 1), (2 \to \mathbb{2}), (\mathbb{P} \to \mathbb{2})\}$
- $(\mathcal{E}_4, \mathcal{F}_4)$ is cofib. generated by $\{(0 \to 1), (2 \to \mathbb{2}), (\mathbb{P} \to \mathbb{2}), (2 \to 1)\}$

where $\mathbb{P} = (\bullet \rightrightarrows \bullet)$.

The right classes are:

$$\mathcal{F}_1 = \{\text{full and surjective-on-objects functors}\}$$
$$\mathcal{F}_2 = \{\text{full and bijective-on-objects functors}\}$$
$$\mathcal{F}_3 = \{\text{fully faithful and surjective-on-objects functors}\}$$
$$\mathcal{F}_4 = \{\text{isos}\}$$

Note that $\mathcal{F}_3$ is the class of trivial fibrations for the canonical model structure on **Cat**.

Four clans for categories

These correspond to the following clans:

$\mathcal{T}_1 = \{\text{free cats on fin. graphs}\}^{\mathsf{op}}$ $\qquad$ $\mathcal{T}_1^\dagger = \{\text{graph inclusions}\}$

$\mathcal{T}_2 = \{\text{free cats on fin. graphs}\}^{\mathsf{op}}$ $\qquad$ $\mathcal{T}_2^\dagger = \{\text{injective-on-edges maps}\}$

$\mathcal{T}_3 = \{\text{f.p. cats}\}^{\mathsf{op}}$ $\qquad$ $\mathcal{T}_3^\dagger = \{\text{injective-on-objects functors}\}$

$\mathcal{T}_4 = \{\text{f.p. cats}\}^{\mathsf{op}}$ $\qquad$ $\mathcal{T}_4^\dagger = \{\text{all functors}\}$

Syntax: four GATs for categories

- Syntactically, adding $(2 \to 1)$ to the generators turns the diagonal of the type $\vdash \; O$ of objects into a display map. This corresponds to adding an extensional identity type with rules

  - $x\,y : O \vdash E(x, y)$ type
  - $x : O \vdash r : E(x, x)$
  - $x\,y : O\,,\, p : E(x, y) \vdash x = y$
  - $x\,y : O\,,\, p\,q : E(x, y) \vdash p = q$

  to the GAT.

- Similarly, adding $(\mathbb{P} \to 2)$ corresponds to adding an extensional identity type with rules

  - $x\,y : O\,,\, f\,g : A(x, y) \vdash F(f, g)$ type
  - $x\,y : O\,,\, f : A(x, y) \vdash s : F(f, f)$
  - $x\,y : O\,,\, f\,g : A(x, y)\,,\, p : F(f, g) \vdash f = g$
  - $x\,y : O\,,\, f\,g : A(x, y)\,,\, p\,q : F(f, g) \vdash p = q$

  to the dependent type $x\,y : O \vdash A(x, y)$ of arrows.

Models in higher types

Models of $\mathcal{T}_1$ in $\mathcal{S}$ are **Segal spaces**, and adding extensional identity types to $\vdash\ O$ or to $x\,y : O\ \vdash\ A(x, y)$ forces the respective types to be 0-truncated. Thus:

$$\infty\text{-}\mathbf{Mod}(\mathcal{T}_1) = \{\text{Segal spaces}\}$$
$$\infty\text{-}\mathbf{Mod}(\mathcal{T}_2) = \{\text{Segal categories}\}$$
$$\infty\text{-}\mathbf{Mod}(\mathcal{T}_3) = \{\text{pre-categories}\}$$
$$\infty\text{-}\mathbf{Mod}(\mathcal{T}_4) = \{\text{discrete 1-categories}\}$$

The Fat Small Object Argument

Recall: Quillen's small object argument

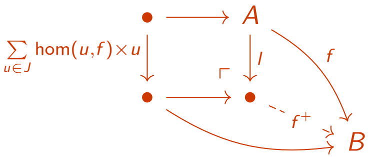
Theorem

Given a small collection $J \subseteq \mathrm{mor}(\mathfrak{X})$ of arrows in a presentable category $\mathfrak{X}$, let

$$\mathcal{R} = \mathsf{RLP}(J) \qquad \text{and} \qquad \mathcal{L} = \mathsf{LLP}(\mathcal{R}).$$

Then $(\mathcal{L}, \mathcal{R})$ is a weak factorization system.

Proof idea: To factor $f : A \to B$, form the pushout



Then $l \in \mathcal{L}$, and we iterate the operation on f^+ transfinitely until the remainder is in $\mathcal{R}$.

Interpretation: Construct fibrant replacement of f in $\mathfrak{X}/B$ by attaching cells until all lifting problems can be solved.

Fat Small Object Argument: Idea

- If the domains of all $u \in J$ are presentable, then every cell attachment factors through a finite stage of the transfinite iteration.
- The FSOA organizes the cell attachments into a 'fatter', and 'shorter' diagram which makes this explicit.
- We present the construction only for the special case factoring $0 \to 1$
- Factoring more general maps $H(\Gamma) \to A$ can be reduced to this case using the following lemmas.

Slicing and coslicing

Slicing lemma

Given a clan $\mathcal{T}$ and $A \in \mathcal{T}\text{-Mod}$, we have $\mathcal{T}\text{-Mod}/A \simeq \underleftarrow{\text{elts}}(A)\text{-Mod}$.

Coslicing lemma

Given a clan $\mathcal{T}$ and $\Gamma \in \mathcal{T}$, we have $H(\Gamma)/\mathcal{T}\text{-Mod} \simeq \mathcal{T}(\Gamma)\text{-Mod}$.

Both equivalences preserve the weak factorization systems.

Finite complexes

Definition

A **finite complex** in a coclan[9] $\mathcal{C}$ is a diagram $D : P \to \mathcal{C}$ where

1. P is a finite poset,
2. $\mathrm{colim}(D_{<x} : P_{<x} \to \mathcal{C})$ exists for all $x \in P$, and the canonical map

$$\alpha_x : \mathrm{colim}(D_{<x}) \to D_x$$

   is a codisplay map, and
3. we have $x = y$ whenever $P_{<x} = P_{<y}$, $D_x = D_y$, and $\alpha_x = \alpha_y : \mathrm{colim}(D_{<x}) \to D_x$.

- One can show that $\mathrm{colim}(D)$ exists for all finite complexes, in particular condition 2 is redundant.
- A finite complex describes a **stratification** of an object in a coclan by/into a **finite set of cell attachments**.
- Condition 3 says that every cell can only be attached once at every stage.

[9] A coclan is the opposite of a clan.

The preorder of finite complexes

Definition

A **morphimsm of finite copmlexes** from $(D : P \to \mathcal{C})$ to $(E : Q \to \mathcal{C})$ is a sieve inclusion $f : D \to E$ such that $E \circ f = D$.

Lemma

The category $\mathsf{FC}(\mathcal{C})$ *of finite complexes in a small coclan* $\mathcal{C}$ *is an essentially small preorder with finite joins.*

The factorization of $0 \to 1$ is now computed as the (filtered) colimit of the composite functor

$$\mathsf{FC}(\mathcal{C}) \xrightarrow{\mathsf{colim}} \mathcal{C} \xrightarrow{H} \mathcal{C}^{\mathsf{op}}\text{-}\mathsf{Mod}.$$

Lemma

The object $C = \mathsf{colim}_{(P,D)\in \mathsf{FC}(\mathcal{C})} H(\mathsf{colim}(D))$ *is a* 0*-extension in* $\mathcal{C}^{\mathsf{op}}\text{-}\mathsf{Mod}$ *and* $C \to 1$ *is full.*

0-*extensions are flat*

Definition

A **flat** algebra over a clan $\mathcal{T}$ is a filtered colimit of hom-algebras $\mathrm{hom}(\Gamma, -)$. Equivalently, an algebra $A \in \mathcal{T}\text{-Mod}$ is flat, if its category of elements $\overleftarrow{\mathrm{elts}}(A)$ is filtered.

Lemma

0-extensions in $\mathcal{T}$-Mod are flat.

Proof.

Let $E \in \mathcal{T}\text{-Mod}$ be a flat algebra. Applying the FSOA in $\mathcal{T}\text{-Mod}/E \simeq \overleftarrow{\mathrm{elts}}(E)\text{-Mod}$, we obtain a full map $f : F \twoheadrightarrow E$ from a 0-extension F which is a filtered colimit of hom-algebras and therefore flat. f splits as a full maps into a 0-extension, and the claim follows since flat algebras are closed under retract. □

Strictness discussion

- Strictness in the definition of finite complexes and moprhisms of finite complexes feels crucial, thus we have to view clans as **strict 1-categories**.

Related work

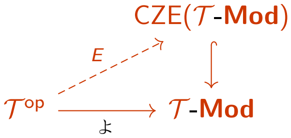
- B. Ahrens, P. North, M. Shulman, and D. Tsementzis. “A higher structure identity principle”. English. In: *Proceedings of the 2020 35th annual ACM/IEEE symposium on logic in computer science, LICS 2020, virtual event, July 8–11, 2020*. New York, NY: Association for Computing Machinery (ACM), 2020
- I. Di Liberti and J. Rosický. “Enriched Locally Generated Categories”. In: (Sept. 2020). arXiv: 2009.10980 [math.CT]
- C.L. Subramaniam. “From dependent type theory to higher algebraic structures”. In: (Oct. 2021). arXiv: 2110.02804 [math.CT]
- S. Henry. “Algebraic models of homotopy types and the homotopy hypothesis”. In: *arXiv preprint arXiv:1609.04622* (2016)

Thanks for your attention!

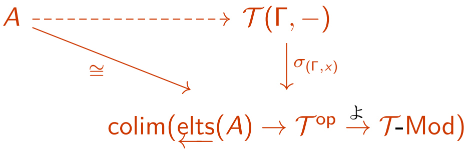
Proof Sketch

Proof sketch: $\mathcal{T}^{\mathrm{op}} \simeq \mathsf{CZE}(\mathcal{T}\text{-}\mathsf{Mod})$

- Easy to see that $\mathcal{T}(\Gamma, -)$ is a compact 0-extension for all $\Gamma \in \mathcal{T}$, thus the Yoneda embedding factors through $\mathsf{CZE}(\mathcal{T}\text{-}\mathbf{Mod})$.

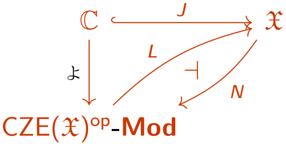


- To see that E is a (Morita) equivalence, it suffices to show that every compact 0-extension is a retract of a hom-algebra $\mathcal{T}(\Gamma, -)$
- This follows from the **fat small object argument**, which implies that $\underleftarrow{\mathrm{elts}}(A)$ is filtered for every 0-extension A — if A is moreover compact, then one of the inclusions of the canonical colimit $A \cong \mathrm{colim}(\underleftarrow{\mathrm{elts}}(A) \to \mathcal{T}^{\mathrm{op}} \xrightarrow{よ} \mathcal{T}\text{-}\mathrm{Mod})$ must split:



Proof sketch: $\mathsf{CZE}(\mathfrak{X})^{\mathsf{op}}\text{-}\mathsf{Mod} \simeq \mathfrak{X}$

- Show that the nerve/realization adjunction



$$L(A) = \mathrm{colim}(\textstyle\int A \to \mathbb{C} \xrightarrow{J} \mathfrak{X})$$
$$N(X) = \mathfrak{X}(J(-), X)$$

  is an equivalence.
- By density the right adjoint N is fully faithful, i.e. the counit is an isomorphism.
- It remains to show that the unit of the adjunction is an isomorphism, i.e.

$$A(C) \xrightarrow{\cong} \mathfrak{X}(C, \mathrm{colim}(\textstyle\int A \to \mathbb{C} \xrightarrow{J} \mathfrak{X})).$$

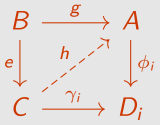
  for all $A \in \mathsf{CZE}(\mathfrak{X})^{\mathsf{op}}\text{-}\mathbf{Mod}$ and $C \in \mathbb{C}$.
- The functor $\mathfrak{X}(C, -)$ preserves filtered colimits and quotients of componentwise-full equivalence relations, so it suffices to decompose $\mathrm{colim}(\int A \to \mathbb{C} \xrightarrow{J} \mathfrak{X})$ in terms of these constructions.
- This is essentially what we're doing in the following, using a Reedy style technique.

Proof sketch: $\mathsf{CZE}(\mathfrak{X})^{\mathsf{op}}\text{-}\mathsf{Mod} \simeq \mathfrak{X}$ – *jointly full cones*

Definition

Let $D : \mathcal{I} \to \mathfrak{X}$ be a diagram in a clan-algebraic category.

A cone (A, ϕ) over D is called **jointly full**, if for every cone (C, γ), extension $e : B \to C$ and map $g : B \to A$ constituting a cone morphism $g : (B, \gamma \circ e) \to (A, \phi)$, there exists a map $h : C \to A$ such that



commutes for all $i \in \mathcal{I}$.

- **Observation:** The cone (A, ϕ) is jointly full iff the canonical map to the limit is full.

Proof sketch: $\mathsf{CZE}(\mathfrak{X})^{\mathsf{op}}\text{-}\mathsf{Mod} \simeq \mathfrak{X}$ *– nice diagrams*

Definition

A **nice diagram** in a clan-algebraic category $\mathfrak{X}$ is a 2-truncated semi-simplicial diagram

$$A_2 \;\begin{array}{c} \xrightarrow{\;d_0\;} \\ \xrightarrow{\;d_1\;} \\ \xrightarrow{\;d_2\;} \end{array}\; A_1 \;\begin{array}{c} \xrightarrow{\;d_0\;} \\ \xrightarrow{\;d_1\;} \end{array}\; A_0$$

where

1. A_0, A_1, and A_2 are 0-extensions, and the maps $d_0, d_1 : A_1 \to A_0$ are full,
2. in the square $\begin{array}{ccc} A_2 & \xrightarrow{d_0} & A_1 \\ {\scriptstyle d_2}\downarrow & & \downarrow{\scriptstyle d_1} \\ A_1 & \xrightarrow{d_0} & A_0 \end{array}$ the span constitutes a jointly full diagram over the cospan,
3. there exists a symmetry map $\begin{array}{ccc} A_1 & \xrightarrow{d_1} & A_0 \\ {\scriptstyle d_0}\downarrow & \searrow^{\sigma} & \uparrow{\scriptstyle d_0} \\ A_0 & \xleftarrow{d_1} & A_1 \end{array}$ making the triangles commute, and
4. there exists a 0-extension $\tilde{A}$ and full maps $f, g : \tilde{A} \twoheadrightarrow A_1$ constituting a jointly full cone over the diagram

$$\begin{array}{ccc} A_1 & & A_1 \\ {\scriptstyle d_0}\downarrow & \overset{d_1}{\searrow}\!\!\!\!\underset{d_0}{\swarrow} & \downarrow{\scriptstyle d_1} \\ A_0 & & A_0 \end{array}.$$

Proof sketch: $\mathrm{CZE}(\mathfrak{X})^{\mathrm{op}}\text{-Mod} \simeq \mathfrak{X}$ — *nice diagrams*

Lemma

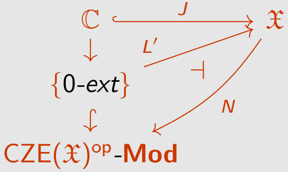
For any nice diagram, the pairing $A_1 \xrightarrow{\langle d_0, d_1 \rangle} A_0 \times A_0$ *admits a decomposition* $A_1 \twoheadrightarrow R \xrightarrow{\langle r_0, r_1 \rangle} A_0 \times A_0$ *into a full map and a monomorphism, and* $\langle r_0, r_1 \rangle$ *is a componentwise-full equivalence relation.*

Lemma

Assume $\mathfrak{X}$ *is clan-algebraic and* $F : \mathfrak{X} \to \mathsf{Set}$ *preserves finite limits and sends full maps to surjections. Then for every nice diagram,* F *preserves coequalizers of the arrows* $d_0, d_1 : A_1 \to A_0$.

Lemma

The restriction L' *of* L *in the nerve/realization adjunction*



to 0*-extensions is fully faithful and preserves full maps and nice diagrams.*

Proof sketch: $\mathsf{CZE}(\mathfrak{X})^{\mathrm{op}}\text{-}\mathsf{Mod} \simeq \mathfrak{X}$ — *Nice diagrams*

Lemma

For every object A of a clan-algebraic category $\mathfrak{X}$ there exists a nice diagram $A_\bullet$ such that

$$A = \operatorname{coeq}(A_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} A_0).$$

Proof.

- A_0 is given by covering A by a 0-extension, i.e. factoring $0 \to A$ as $0 \hookrightarrow A_0 \overset{e}{\twoheadrightarrow} A$.
- A_1 is given by covering the kernel of $A_0 \twoheadrightarrow A$ by a 0-extension

$$\begin{array}{ccccccc} 0 & \hookrightarrow & A_1 & \twoheadrightarrow & R & \xrightarrow{r_0} & A_0 \\ & & & & {\scriptstyle r_1}\downarrow & \lrcorner & \downarrow{\scriptstyle e} \\ & & & & A_0 & \xrightarrow{e} & A \end{array}$$

- A_2 is given by covering the following pullback:

$$\begin{array}{ccccccc} 0 & \hookrightarrow & A_2 & \twoheadrightarrow & \bullet & \longrightarrow & A_1 \\ & & & & \downarrow & \lrcorner & \downarrow{\scriptstyle d_0} \\ & & & & A_1 & \xrightarrow{d_1} & A_0 \end{array}$$

□

Remark: The construction of $A_\bullet$ is a Reedy-style factorization of the maps $0 \to \Delta(A)$ in 2-truncated semi-simplicial objects.

Proof sketch: $\mathsf{CZE}(\mathfrak{X})^{\mathsf{op}}\text{-}\mathsf{Mod} \simeq \mathfrak{X}$ – *the calculation*

Have to show that $AC \cong \mathfrak{X}(C, LA)$ for all $A \in \mathsf{CZE}(\mathfrak{X})^{\mathsf{op}}\text{-}\mathbf{Mod}$ and $C \in \mathsf{CZE}(\mathfrak{X})$. Let $A_\bullet$ be a nice diagram with coequalizer A. We have

$$
\begin{aligned}
\mathfrak{X}(C, LA) &= \mathfrak{X}(C, L(\mathsf{coeq}(A_1 \rightrightarrows A_0))) && \text{since } A = \mathsf{coeq}(A_1 \rightrightarrows A_0) \\
&\cong \mathfrak{X}(C, \mathsf{coeq}(LA_1 \rightrightarrows LA_0)) && \text{since } L \text{ preserves colimits} \\
&\cong \mathsf{coeq}(\mathfrak{X}(C, LA_1) \rightrightarrows \mathfrak{X}(C, LA_0)) && \text{since } \mathfrak{X}(C, -) \text{ preserves coeqs of nice diags} \\
&\cong \mathsf{coeq}(A_1 C \rightrightarrows A_0 C) && \text{since } LA_i = \mathsf{colim}(\textstyle\int A_i \to \mathbb{C} \to \mathfrak{X}) \text{ filtered} \\
&\cong \mathsf{coeq}(\hom(よ(C), A_1) \rightrightarrows \hom(よ(C), A_0)) \\
&\cong \hom(よ(C), \mathsf{coeq}(A_1 \rightrightarrows A_0)) \\
&\cong \hom(よ(C), A) \\
&\cong AC
\end{aligned}
$$