Colimits in the category of pointed types

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Overview

- The ∞-category $\mathcal{U}^*$ of pointed types and *pointed* functions $A \to_* B :\equiv \sum_{f:A\to B} f(a_0) = b_0$ is a useful setting for synthetic homotopy theory.
- Examples include type-theoretic Brown representability (in progress) and the adjunctions

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  between endofunctors of $\mathcal{U}^*$.
- In this talk, the construction of an arbitrary (homotopy) colimit in $\mathcal{U}^*$ as the quotient of a quotient by a quotient.

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  Immediate corollary: *The forgetful functor $\mathcal{U}^* \to \mathcal{U}$* ***creates colimits over*** *Γ if and only if Γ is a* ***tree****.*

Graphs

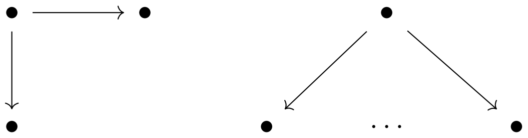
Consider a graph $\Gamma := (\Gamma_0, \Gamma_1)$.

Γ_0 the type of vertices, $\Gamma_1(x, y)$ the type of edges from x to y.

Definition

We say that Γ is a *tree* if the quotient ${}^{\Gamma_0}\!/_{\Gamma_1}$ is contractible.

Examples of trees:



$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots$$

$$\cdots \longrightarrow -2 \longrightarrow -1 \longrightarrow 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots$$

Non-set example (taken from Buchholtz et al. (2023)):

We have a 2-HIT $\mathrm{B}H$ such that

- $\mathrm{B}H$ has fundamental group the *Higman group*
- $\Sigma(\mathrm{B}H) = 1$.

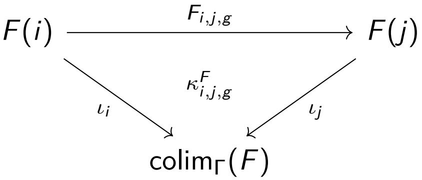
Take

$$\Gamma_0 := \mathbf{bool}$$
$$\Gamma_1(0, 1) := \mathrm{B}H.$$

Colimits

Consider a diagram F of types and functions over Γ.

The 1-HIT $\mathrm{colim}_\Gamma(F)$ is generated by a cocone



under F in the unpointed category $\mathcal{U}$.

Suppose that F is equipped with

- a basepoint b_i of $F(i)$ for each $i : \Gamma_0$
- an identity $p_{i,j,g} : F_{i,j,g}(b_i) = b_j$ for all $i, j : \Gamma_0$ and $g : \Gamma_1(i, j)$.

Form the pushout square

$$\begin{array}{ccc} \operatorname{colim}_\Gamma \mathbf{1} & \xrightarrow{\varphi} & \operatorname{colim}_\Gamma(F) \\ \Big\downarrow & \text{glue} & \Big\downarrow \text{inr} \\ 1 & \xrightarrow[\text{inl}]{} & P \end{array} .$$

We want to exhibit P as the colimit of F in $\mathcal{U}^*$.

In particular, we want an identity

$$
\begin{array}{ccc}
(F(i), b_i) & \xrightarrow{(F_{i,j,g},\, p_{i,j,g})} & (F(j), b_j) \\
& \searrow^{\left(\mathsf{inr}\circ\iota_i,\, \mathsf{glue}(\iota_i^{\mathbf{1}}(*))^{-1}\right)} \quad {}^{\left(\mathsf{inr}\circ\iota_j,\, \mathsf{glue}(\iota_j^{\mathbf{1}}(*))^{-1}\right)}\swarrow & \\
& (P, \mathsf{inl}(*)) &
\end{array}
$$

of pointed maps.

We have a dependent path $C_{i,j,g}$

$$
\mathsf{ap}_{\mathsf{inr}}(\kappa^F_{i,j,g}(b_i))^{-1} \cdot \mathsf{ap}_{\mathsf{inr}\circ\iota_j}(p_{i,j,g}) \cdot \mathsf{glue}(\iota_j^{\mathbf{1}}(*))^{-1}
$$
$$
=
$$
$$
\left(\left(\kappa^{\mathbf{1}}_{i,j,g}\right)_* (\mathsf{glue}(\iota_j^{\mathbf{1}}(*)))\right)^{-1}
$$
$$
=
$$
$$
\mathsf{glue}(\iota_i^{\mathbf{1}}(*))^{-1}
$$

from the pointedness proof of $(\mathsf{inr} \circ \iota_j)^* \circ F^*_{i,j,g}$ to that of $(\mathsf{inr} \circ \iota_i)^*$ over the homotopy $\mathsf{ap}_{\mathsf{inr}\circ -}(\kappa^F_{i,j,g})$ between underlying functions.

Then the data

$$\kappa^P := \left(\left(\lambda i. \left(\mathsf{inr} \circ \iota_i, \mathsf{glue}(\iota_i^{\mathbf{1}}(*))^{-1}\right)\right), \lambda j \lambda i \lambda g. \left(\mathsf{ap}_{\mathsf{inr}\circ -}(\kappa^F_{i,j,g}), C_{i,j,g}\right)\right)$$

equips the pointed type $(P, \mathsf{inl}(*))$ with the structure of a cocone under F in the *pointed* category $\mathcal{U}^*$.

Theorem

The post-composition map

$$(P \to_* T) \to \lim_{i:\Gamma^{\mathrm{op}}}(F(i) \to_* T)$$

is an equivalence for every pointed type T, i.e., $\left(P, \kappa^P\right)$ is a colimiting pointed cocone under F.

A colimit of F in $\mathcal{U}^*$ can be postulated as a non-recursive 2-HIT K.

Constructors endow K with pointed cocone structure under F.

By van Doorn et al. (2017), K can be constructed, roughly, by quotienting a quotient by a family of circles.

This construction has the “wrong” form for our application, but it’s equivalent to our construction.

If ${}^{\Gamma_0}\!/\!_{\Gamma_1}$ is contractible at, say, g, then we have a pointed cocone under F

$$\begin{array}{ccc} F(i) & \xrightarrow{\quad (F_{i,j,g},\, p_{i,j,g}) \quad} & F(j) \\ & {\scriptstyle \iota_i^*} \searrow \qquad \swarrow {\scriptstyle \iota_j^*} & \\ & (\mathrm{colim}_\Gamma(F), \varphi(g)) & \end{array}$$

whose homotopy between underlying functions is precisely $\kappa^F_{i,j,g}$.

In this case, $\mathrm{colim}_\Gamma(F) \xrightarrow{\mathrm{inr}} P$ induces an equivalence of pointed cocones.

Corollary

The forgetful functor $\mathcal{U}^* \to \mathcal{U}$ creates all colimits over Γ if and only if Γ is a tree.

Applications

- *Every moderately nice functor* $(\mathcal{U}^*)^{\mathsf{op}} \to \mathbf{Set}$ *is representable on a subuniverse consisting of iterated pointed colimits of nice spaces.*

  We know that internally, such a subuniverse will include many familiar spaces.

- Left adjoints such as $\Sigma(-)$ and $- \wedge A$ preserve colimits of tree-indexed diagrams of pointed types and maps.

Future work

- Formalize main theorems in Agda.
- Find / learn of new applications.

References

Ulrik Buchholtz, Tom de Jong and Egbert Rijke, *On epimorphisms and acyclic types in HoTT*

Floris van Doorn, Jakob von Raumer and Ulrik Buchholtz, *Homotopy Type Theory in Lean*