Goursat Homology

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Outline

Coherent Functors

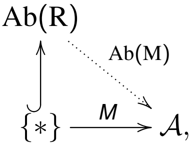
- R - an associative ring;
- $\mathrm{R\text{-}mod}$ - the category of finitely presented left R-modules;
- $(\mathrm{R\text{-}mod}, \mathrm{Ab})$ - the category of Ab-valued additive functors;
- $(R, -) := \mathrm{Hom}_R(R, -)$ - the forgetful functor.

Theorem (M. Auslander, 1965)

The category $\mathrm{fp}(\mathrm{R\text{-}mod}, \mathrm{Ab})$ is abelian.

The Free Abelian Category over *R*

- $\{*\}$ - the one-object preadditive category with $\mathrm{End}(*) = R$;
- the *free abelian category* over R is an inclusion



such that every additive $M\colon \{*\} \to \mathcal{A}$ extends to an exact functor $\mathrm{Ab(M)}$, as indicated.

Theorem (P. Freyd, 1966)

$\mathrm{Ab(R)} \to \mathrm{fp(R\text{-}mod, Ab)},\ * \mapsto \mathrm{(R, -)}$, is an equivalence.

The Lattice of Subfunctors

Definition

$\mathbb{L}(R, n) := \mathrm{Sub}_{\mathrm{Ab(R)}}(R, -)^n$ is the modular lattice of subfunctors of the n-th power of the forgetful functor.

Consider a free presentation of $M \in \mathrm{R\text{-}mod}$,

$$ {}_R R^m \xrightarrow{-\times A} {}_R R^n \longrightarrow {}_R M \longrightarrow 0, $$

where $A = (r_{ij})$ is an $m \times n$ matrix over R. In $\mathrm{Ab(R)}$,

$$ 0 \longrightarrow (M, -) \longrightarrow (R, -)^n \xrightarrow{A\times -} (R, -)^m. $$

Outline

Definition

- The language for left R-modules is given by

$$\mathcal{L}(R) = \mathcal{L}(+, -, 0, r)_{r \in R}.$$

- A *positive primitive* formula in $\mathcal{L}(R)$ is of the form

$$\begin{aligned} \varphi(u_1, \ldots, u_n) &= \exists v_1, \ldots, v_k \; (A\mathbf{u}^t \doteq B\mathbf{v}^t) \\ &= B \mid A\mathbf{u}^t \end{aligned}$$

- Every $\varphi \in \mathbb{L}(R, n)$ is of the form

$$\varphi(-)\colon M \mapsto \varphi(M) = \{\mathbf{b} \in M^n \mid \exists \mathbf{c} \in M^k \; (A\mathbf{b}^t = B\mathbf{c}^t)\} \subseteq M^n.$$

- If $F \in \mathrm{Ab(R)}$ then there are $\psi \leq \varphi$ in some $\mathbb{L}(R, n)$ such that $F \cong \varphi/\psi$.

Historical Remarks

- For a division ring Δ, $\mathbb{L}(\Delta, n) \setminus \{0\}$ is better known as *n-dimensional projective geometry* over Δ.

The Fundamental Theorem of Projective Geometry

If $\mathbb{L}(\Delta, m) \cong \mathbb{L}(\Delta', n)$ and $m \geq 3$, then $m = n$ and $\Delta \cong \Delta'$.

- $\mathbb{L}(R, n) \cong \mathbb{L}(M_n(R), 1)$
- $\mathbb{L}(R, 1)$ is complemented iff R is *von Neumann regular,* i.e. $R \models \forall x \exists y\ (xyx = x)$.

Operations on Positive Primitive Formulae

The Face Operations

- $f_i \colon \varphi(u_1, \dots, u_n) \mapsto \varphi(u_1, \dots, u_{i-1}, 0, u_i, \dots, u_{n-1})$
- $b_i \colon \varphi(u_1, \dots, u_n) \mapsto \exists u\, \varphi(u_1, \dots, u_{i-1}, u, u_i, \dots, u_{n-1})$

There are also two degeneracy operations:

- $s_i^+ \colon \varphi(u_1, \dots, u_n) \mapsto \varphi(u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_{n+1})$
- $s_i^- \colon \varphi(u_1, \dots, u_n) \mapsto \varphi(u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_{n+1}) \wedge u_i \doteq 0$

- (b_i, s_i^+) and (s_i^-, f_i) are adjoint pairs.

Outline

Goursat's Lemma

Goursat's Lemma (1889)

Let G_1 and G_2 be groups and $\Gamma \leq G_1 \times G_2$. Then

$$\frac{\pi_1(\Gamma)}{\Gamma \cap (G_1 \times 1)} \cong \frac{\pi_2(\Gamma)}{\Gamma \cap (1 \times G_2)}.$$

In $\mathrm{Ab}(\mathrm{R})$

If $\varphi(u, v) \in \mathbb{L}(R, 2)$, then

$$\frac{\exists v\ \varphi(u, v)}{\varphi(u, 0)} \cong \frac{\exists u\ \varphi(u, v)}{\varphi(0, v)}.$$

The Goursat Chain Complex

Define

- $G_n(R) = \oplus_{\varphi \in \mathbb{L}(R,n+1)} \mathbb{Z}[\varphi]$, for $n \geq 0$;
- $G_{-1}(R) = \mathbb{Z}$; and
- $G_n(R) = 0$, for $n < -1$,

with boundary map $d_n \colon [\varphi] \mapsto \sum_i (-1)^i([b_i(\varphi)] - [f_i(\varphi)])$.

To calculate $H_0(R)$, consider

$$G_1(R) \xrightarrow{d_1} G_0(R) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

0-Dimensional Goursat Homology

Theorem (IH)

$H_0(R) \cong K_0(\mathrm{Ab(R)})$

Proof:

- Every $F \in \mathrm{Ab(R)}$ has a filtration

$$0 = F_0 \leq F_1 \leq F_2 \leq \cdots \leq F_n = F$$

  such that $F_{i+1}/F_i \cong \varphi_i/\psi_i$ with $\psi_i \leq \varphi_i \in \mathbb{L}(R, 1)$.
- The map $K_0(\mathrm{Ab(R)}) \to \mathrm{H_0(R)}$, $[F] \mapsto \sum_i ([\varphi_i] - [\psi_i])$ is well-defined, by the Schreier Refinement Theorem.

REFERENCES:

1. Artin, Emil, *Geometric Algebra,* Dover, 2016.
2. Auslander, Maurice, Coherent Functors, *Proceedings of the Conference on Categorical Algebra at La Jolla 1965,* Springer-Verlag, 1966.
3. Freyd, Peter, Representations of Abelian Categories, *ibid.*
4. Goursat, Édouard, Sur les substitutions régulières de l'espace, *Annales Scientifiques de l'École Normale Supérieure* **6,** 9-102, 1889.
5. Herzog, Ivo, Linear Algebra over a Ring, arXiv:0909.0436.

5 Posur, Sebastian, A Constructive Approach to Freyd Categories, Applied Categorical Structures **29,** 171-211, 2021.