A Foundation for Synthetic Algebraic Geometry

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Homotopy Type Theory 2023

Thinking about algebra in a geometric way

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But does $\operatorname{Spec}(A)$ retain all information from A? No. :-(

Classical vs synthetic

How can we make the functor

$$A \mapsto \mathrm{Spec}(A)$$

fully faithful?

Classical algebraic geometry	Synthetic algebraic geometry
Endow $\mathrm{Spec}(A)$ with additional structure: ▶ Zariski topology ▶ structure sheaf $\mathcal{O}_{\mathrm{Spec}(A)}$	Just postulate it! :-) **Axiom (SQC)[1].** The map $$A \to R^{\mathrm{Spec}\,A}$$ $$a \mapsto (\varphi \mapsto \varphi(a))$$ is an equivalence, for every finitely presented R-algebra A.

[1]"Synthetic Quasi-Coherence", due to Ingo Blechschmidt

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- $\operatorname{Spec}(R/(r)) = (r = 0)$. Thus: if $r \neq 0$, then r is invertible.
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Axiom: The ring R is local.

- If $r_1, \ldots, r_n : R$ are not all zero, then some r_i is invertible.

Closed and open propositions

For $r_1, \dots, r_n : R$ we have the propositions

$$V(r_1, \dots, r_n) :\equiv (r_1 = \dots = r_n = 0),$$

$$D(r_1, \dots, r_n) :\equiv (r_1 \text{ inv.} \vee \dots \vee r_1 \text{ inv.}).$$

Then define:

$$\text{closedProp} :\equiv \sum_{p:\text{hProp}} \exists r_1, \dots, r_n . (p = V(r_1, \dots, r_n))$$

$$\text{openProp} :\equiv \sum_{p:\text{hProp}} \exists r_1, \dots, r_n . (p = D(r_1, \dots, r_n))$$

A *closed subtype* of X is a map $X \to \text{closedProp}$.
An *open subtype* of X is a map $X \to \text{openProp}$.

Schemes

A type X is an *affine scheme* if it is of the form $X = \mathrm{Spec}(A)$.

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Example. *Projective n-space*:

$$\begin{aligned}\mathbb{P}^n &:= \{\,\text{lines through } 0 \text{ in } R^{n+1}\,\}\\ &:= \{\,\text{sub-}R\text{-modules } L \subseteq R^{n+1} \text{ such that } \|L = R^1\|\,\}\end{aligned}$$

is a scheme:

$$U_i(L) := (b_i \text{ is invertible}) \ \textit{(for any chosen base } \{b\} \textit{ of } L\textit{)}$$

Line bundles

The type

$$\text{Lines} := \sum_{L:R\text{-Mod}} \|L = R^1\|$$

has a wild group structure:

- $L \otimes L'$ is again a line
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The *Picard group* of X is

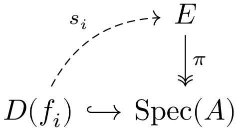
$$\mathrm{Pic}(X) := \| X \to \mathrm{Lines} \|_{\mathsf{set}}.$$

(In fact, $\mathrm{Lines} = K(R^\times, 1)$ and $\mathrm{Pic}(X) = H^1(X, R^\times)$.)

Zariski-local choice

For $f : A$ define $D(f) := \{\, \varphi : \mathrm{Spec}(A) \mid \varphi(f) \text{ is invertible} \,\}$.

Axiom (Zariski-local choice):
For every surjective π, there merely exist local sections s_i



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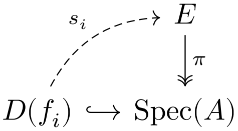

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Some consequences:

- Every line bundle (on a scheme) is locally trivial.
- $(\mathrm{Spec}\, A \to \mathrm{closedProp}) \cong \{$ fin. gen. ideals in $A \,\}$
- $(\mathrm{Spec}\, A \to \mathrm{openProp}) \cong \{$ fin. gen. radical ideals in $A \,\}$
- If $U \subseteq X$ open and $V \subseteq U$ open, then $V \subseteq X$ open.

The scheme classifier

Let $\mathrm{Sch} \hookrightarrow \mathrm{Type}$ be the type of schemes.

Theorem. Let X be a scheme and $Y : X \to \mathrm{Sch}$ be given. Then $\sum_{x:X} Y(x)$ is a scheme.

Corollary. For $f : Y \to X$ a map between schemes and $x : X$, the fiber $\sum_{y:Y} \underbrace{f(y) = x}_{\text{a scheme}}$ is a scheme.

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This means we have a scheme classifier:

$$\mathrm{Sch}/X = (X \to \mathrm{Sch})$$

In particular, we have a subscheme-classifier: $\mathrm{Sch} \cap \mathrm{hProp}$.

Thank you!