# $\pi_4\mathbb{S}^3 \not\simeq 1$ and another Brunerie number in CCHM

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# Thanks

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- organizers

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- Marcin Jan Grzybowski for ongoing work on visualizations `https://github.com/marcinjangrzybowski/cubeViz2`

# Pictures from cubeViz2 (by Marcin Jan Grzybowski)

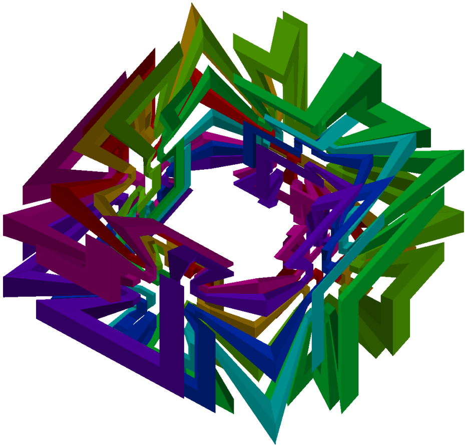

Figure: "Twisting" Hopf fibration (generator of $\pi_3\mathbb{S}^2$)

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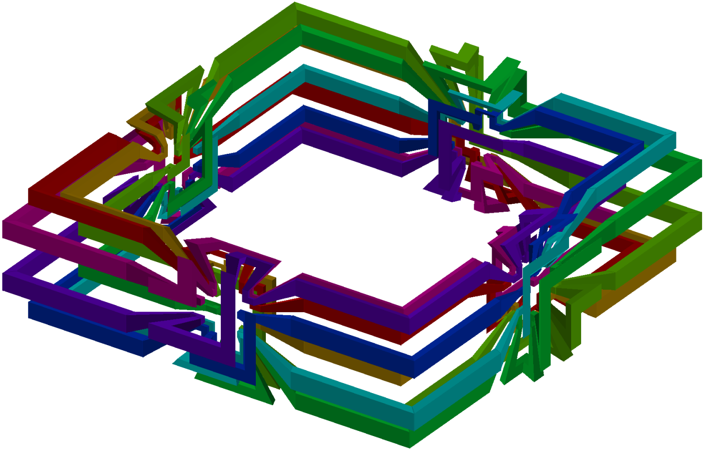

Figure: "Twisting" Hopf fibration again

# Disclaimer

- All results in `cubicaltt` implementation of CCHM cubical type theory (Cohen, Coquand, Huber, Mörtberg 2015 [3][4], Coquand, Huber, Mörtberg 2018 [5]) unless otherwise noted (Cohen, Coquand, Huber, Mörtberg 2015 [3], Coquand, Huber, Mörtberg 2018 [5])

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- No suspensions, joins, pushouts, colimits...
  - e.g. $\mathbb{S}^n$ is the point and n-loop HIT

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- and in Kovacs' cctt:

```
Normal form of brunerie:

pos (suc (suc zero))

Normalized in 0.102955318s
```

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- short proof of syllepsis and "the generator of $\pi_4\mathbb{S}^3$ has order 2"
- another proof of $\pi_4\mathbb{S}^3 \cong \mathbb{Z}/2\mathbb{Z}$? (as suggested by Snyder, Ljungström)

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conclusion
  : Path Ω⁴𝕊³ genPi4s3 refl ->
    Path bool true false
  = cong computeΩ⁴𝕊³
```

# Pontryagin's Theorem

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- First proof of $\pi_4\mathbb{S}^3 \cong \mathbb{Z}/2\mathbb{Z}$!

# The Brunerie number is 2

- Brunerie's recipe: define an isomorphism $\pi_3\mathbb{S}^2 \to \mathbb{Z}$, plug in the Whitehead product $[i_2, i_2] : \pi_3\mathbb{S}^2$.

# Whitehead products

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- So we consider "cubical $(\alpha, \beta)$-extensions":

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# Whitehead products

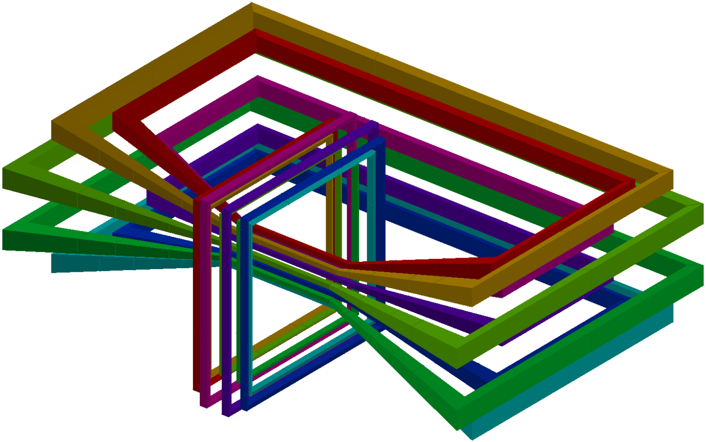

Figure: $[\alpha, \beta] : \Omega^3 A$ (from Grzybowski's cubeViz2)

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- and $(\alpha, \beta)$-extension $\simeq ([\alpha, \beta] =_{\Omega^3 A} \mathsf{refl})$

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- James construction $\Omega\mathbb{S}^2 \simeq \mathsf{J}\mathbb{S}^1$

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- recursion principle: $\mathsf{recJ}\mathbb{S}^1 : A \to (\Pi_{x:A}\Omega(A,x)) \to \mathsf{J}\mathbb{S}^1 \to A$

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- Local-global looping (Kraus, Sattler 2015 [6]) aka "the key maneuver [for $\pi_n \mathbb{S}^n$]" (Licata, Brunerie 2013 [7])

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- Plug in the Whitehead product $[\mathrm{surf}, \mathrm{surf}]$, it computes to 2 (in `cubicaltt` or `cctt`)

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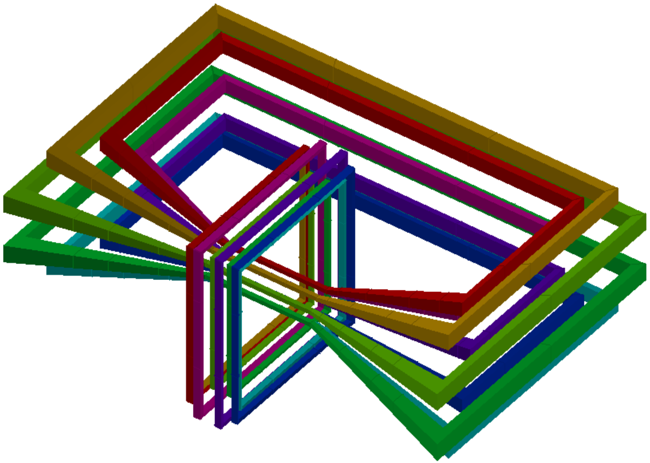

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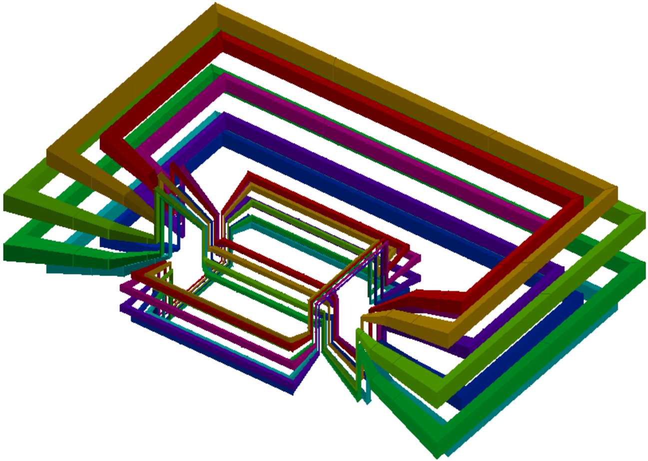

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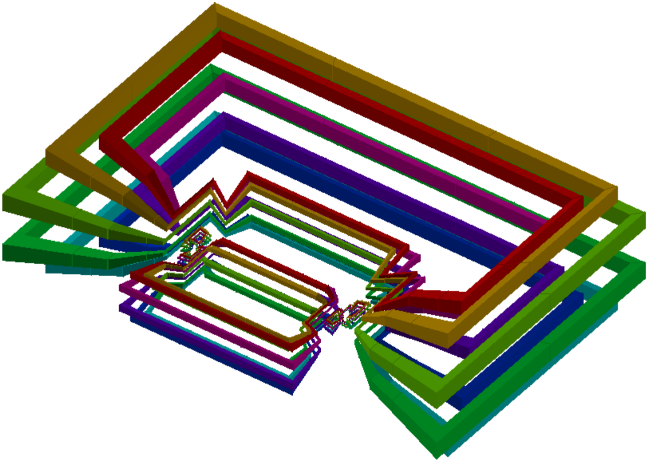

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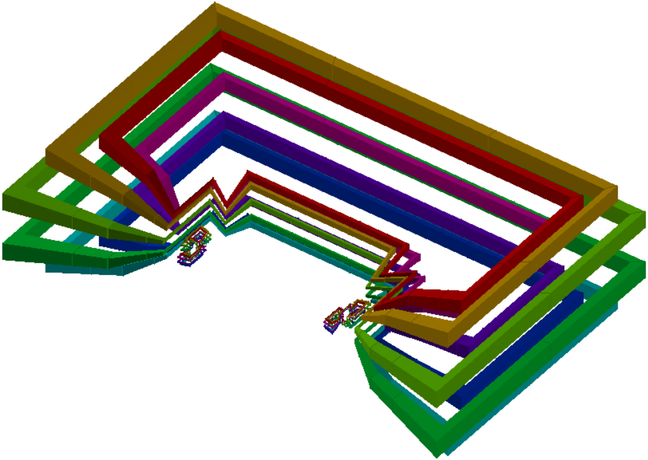

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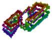

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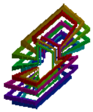

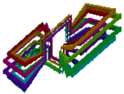

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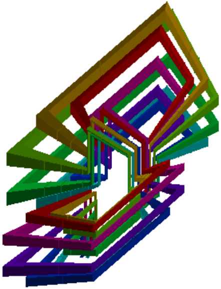

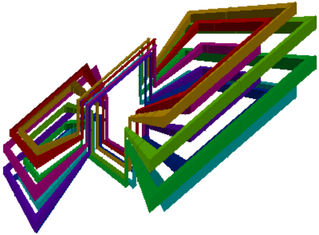

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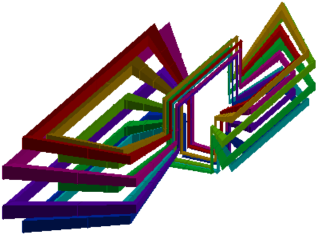

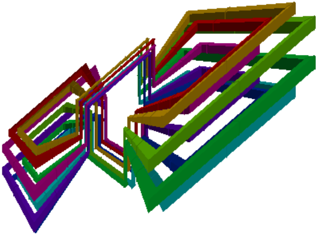

# Syllepses are trivializations of Whitehead products

$$\Pi_{p,q:\Omega^2 A}\ (\mathsf{EH}(p,q) = \mathsf{EH}(q,p)^{\mathsf{T}}) \simeq ([p,q] =_{\Omega^3 A} \mathsf{refl})$$

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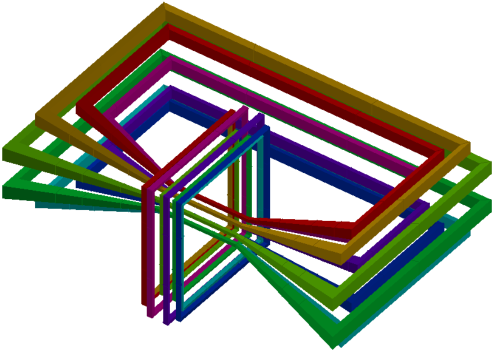

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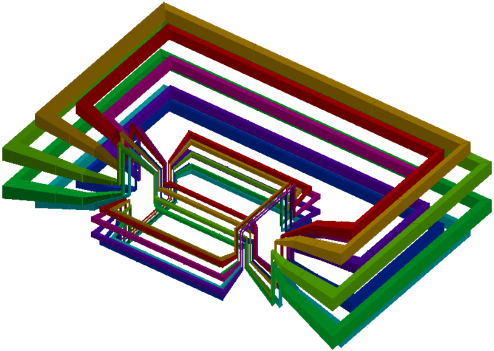

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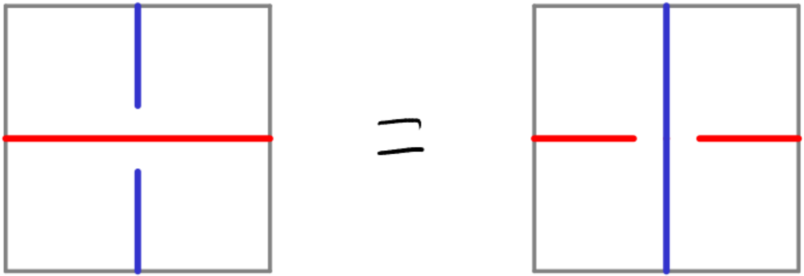

Figure: Syllepses

# Syllepses are trivializations of Whitehead products

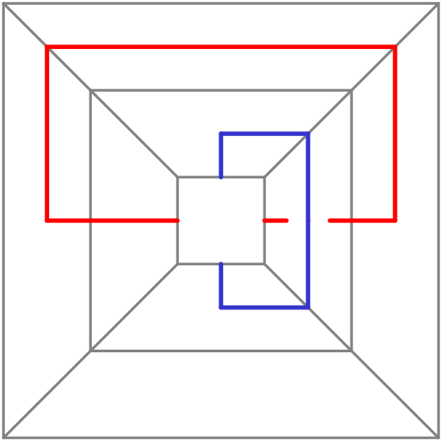

Figure: a funny tube

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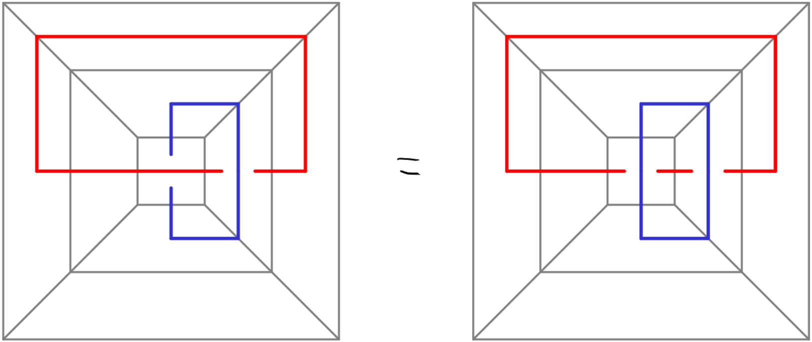

Figure: funny syllepses

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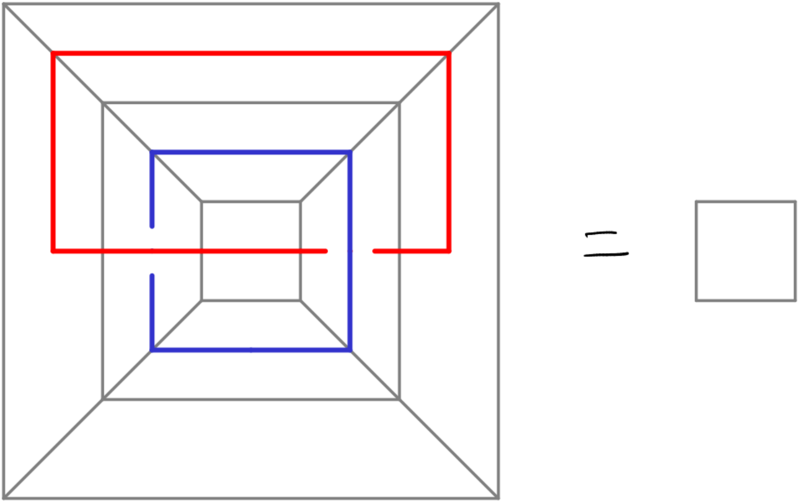

Figure: trivializations of the Whitehead product!

# $\pi_4 \mathbb{S}^3 \not\simeq 1$

Idea:

$$\pi_4 \mathbb{S}^3 \to \pi_3 \mathsf{J}_2 \mathbb{S}^2 \to \pi_2 \left(\mathbb{S}^1 \times \|\mathbb{S}^2/2\|_2\right) \to \pi_2(\mathbb{S}^2/2) \to \mathsf{bool}$$

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This is not what I actually did, but it's easier to explain. At least one very ugly thing does work in `cubicaltt`...

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```
data J₂𝕊² : Type where
  base : J₂𝕊²
  surf : Ω²(J₂𝕊², base)
  syll : (surf, surf)-extension
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  surf : Ω²(J₂𝕊², base)
  syll : EH(surf, surf) = EH(surf, surf)ᵀ
```

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- Crux: $\pi_3 \mathsf{J}_2 \mathbb{S}^2 \to \pi_2 \left(\mathbb{S}^1 \times \|\mathbb{S}^2/2\|_2\right)$

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- Want: $\Omega \mathsf{J}_2 \mathbb{S}^2 \to \left( \mathbb{S}^1 \times \| \mathbb{S}^2 / 2 \|_2 \right)$

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- Want: $\alpha : \Omega^2(\mathcal{U}, \mathbb{S}^1 \times \|\mathbb{S}^2/2\|_2)$ with a syllepsis $\mathsf{EH}(\alpha, \alpha) = \mathsf{EH}(\alpha, \alpha)^\intercal$

# $\pi_4 \mathbb{S}^3 \not\simeq 1$

- 2+2 conjecture:

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- "Syllepsis in the universe" conjecture:

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- Given $a, b : \Pi_{x:A} \Omega(A, x)$, define $\alpha, \beta : \Omega^2(\mathcal{U}, A)$ by $\alpha := \mathsf{global}(a), \beta := \mathsf{global}(b)$.

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- Then, syllepses $\mathsf{EH}(\alpha, \beta) = \mathsf{EH}(\beta, \alpha)^\mathsf{T}$ are equivalent to $\Pi_{x:A} \mathsf{apd}_a(b(x)) = \mathsf{apd}_b(a(x))^\mathsf{T}$.

# $\pi_4\mathbb{S}^3 \not\simeq 1$

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- Then, syllepses $\mathsf{EH}(\alpha, \beta) = \mathsf{EH}(\beta, \alpha)^\intercal$ are equivalent to $\Pi_{x:A}\mathsf{apd}_a(b(x)) = \mathsf{apd}_b(a(x))^\intercal$.
- Maybe we can prove it using Baker's insight?

# $\pi_4 \mathbb{S}^3 \not\simeq 1$

- If the "syllepsis in the universe" conjecture holds, we should be able to use it to construct the $\alpha : \Omega^2(\mathcal{U}, \mathbb{S}^1 \times \|\mathbb{S}^2/2\|_2)$ with a syllepsis $\mathsf{EH}(\alpha, \alpha) = \mathsf{EH}(\alpha, \alpha)^{\mathsf{T}}$

# $\pi_4\mathbb{S}^3 \not\simeq 1$

- If the "syllepsis in the universe" conjecture holds, we should be able to use it to construct the $\alpha : \Omega^2(\mathcal{U}, \mathbb{S}^1 \times \|\mathbb{S}^2/2\|_2)$ with a syllepsis $\mathsf{EH}(\alpha,\alpha) = \mathsf{EH}(\alpha,\alpha)^\intercal$
- The resulting map $\pi_4\mathbb{S}^3 \to$ bool should then compute nontrivially, at least in `cubicaltt`...

# Etc

- $J\mathbb{S}^1$ as a "holographic" model of $\Omega\mathbb{S}^2$:
  - link diagrams
  - $J_3\mathbb{S}^1$: Reidemeister III
  - $J_4\mathbb{S}^1$: Zamolodchikov tetrahedron equation?
  - generalized Pontryagin's theorem w/ "regular dual stratifications" per Christopher Dorn 2023, "An Invitation to Geometric Higher Categories" (???)
- Adams-Hilton construction (see Carlsson, Milgram 1995)
  - for $\Omega J_2\mathbb{S}^2$: $J_3\mathbb{S}^1$ but we can change the signs of crossings
  - Computational univalence allows computing the boundary of the Adams-Hilton cells, in theory???

Thanks!

# References I


Guillaume Brunerie.
*On the homotopy groups of spheres in homotopy type theory*.
PhD dissertation, Université Nice Sophia Antipolis, 2016.
https://arxiv.org/abs/1606.05916.

Ulrik Buchholtz, J Daniel Christensen, Jarl G Taxerås Flaten, and Egbert Rijke.
Central h-spaces and banded types.
*arXiv preprint arXiv:2301.02636*, 2023.

Cyril Cohen, Thierry Coquand, Simon Huber, and Anders Mörtberg.
Cubical type theory: A constructive interpretation of the univalence axiom.
In Tarmo Uustalu, editor, *21st International Conference on Types for Proofs and Programs, TYPES 2015, May 18-21,*

# References II

*2015, Tallinn, Estonia*, volume 69 of *LIPIcs*, pages 5:1–5:34. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2015.

Cyril Cohen, Thierry Coquand, Simon Huber, and Anders Mörtberg.
cubicaltt: Experimental implementation of Cubical Type Theory.
https://github.com/mortberg/cubicaltt, 2015.

Thierry Coquand, Simon Huber, and Anders Mörtberg.
On higher inductive types in cubical type theory.
In *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 255–264, 2018.

Nicolai Kraus and Christian Sattler.
Higher homotopies in a hierarchy of univalent universes.
*ACM Transactions on Computational Logic (TOCL)*, 16(2):1–12, 2015.

# References III


Daniel R Licata and Guillaume Brunerie.
$\pi$ n (s n) in homotopy type theory.
In *Certified Programs and Proofs: Third International Conference, CPP 2013, Melbourne, VIC, Australia, December 11-13, 2013, Proceedings 3*, pages 1–16. Springer, 2013.

Lev Pontryagin.
Classification of continuous maps of a complex into a sphere, Communication I.
*Doklady Akademii Nauk SSSR*, 19(3):147–149, 1938.