Efficient Evaluation for Cubical Type Theories

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Many more definitions to go!

Outline

1. Normalization-by-evaluation for MLTT
2. NbE for CTT
3. Implementation & benchmarks
4. Conclusions

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Solution

- Separate *syntax* (program code) from *semantic values* (runtime objects).
- The syntax only supports *evaluation* into values.
- Values support efficient β-reduction, without using recursive substitution.

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I focus on a *practical flavor* of NbE which has several differences to the nicest *formal* NbE.

Informal NbE (1)

We omit types of things for brevity.

Syntax & values

$$\begin{aligned} \Gamma, \Delta &: \mathsf{Con} & \sigma, \delta &: \mathsf{Env}\,\Gamma\,\Delta \\ t, u &: \mathsf{Tm}\,\Gamma & v &: \mathsf{Val}\,\Gamma \\ \sigma, \delta &: \mathsf{Sub}\,\Gamma\,\Delta \end{aligned}$$

Operations

$$\begin{aligned} \mathsf{eval} &: \mathsf{Env}\,\Gamma\,\Delta \to \mathsf{Tm}\,\Delta \to \mathsf{Val}\,\Gamma \\ \mathsf{quote} &: \mathsf{Val}\,\Gamma \to \mathsf{Tm}\,\Gamma \\ \mathsf{conv} &: \mathsf{Val}\,\Gamma \to \mathsf{Val}\,\Gamma \to \mathsf{Bool} \end{aligned}$$

$\mathsf{Val}\,\Gamma$ has the same structure as $\mathsf{Tm}\,\Gamma$, except each binder is replaced with a **closure**. A closure stores a variable name x, an environment $\sigma : \mathsf{Env}\,\Gamma\,\Delta$ and a $t : \mathsf{Tm}\,(\Delta, x)$.

Informal NbE (2)

$$
\begin{aligned}
&\mathsf{eval} : \mathsf{Env}\,\Gamma\,\Delta \to \mathsf{Tm}\,\Delta \to \mathsf{Val}\,\Gamma \\
&\mathsf{eval}\,\sigma\,x \quad\; :\equiv \sigma\,x \\
&\mathsf{eval}\,\sigma\,(\lambda\,x.\,t) :\equiv \lambda_{\mathsf{Val}}\,(x,\ \sigma,\ t) \\
&\mathsf{eval}\,\sigma\,(t\,u) \quad :\equiv \mathsf{case}\ \mathsf{eval}\,\sigma\,t\ \mathsf{of} \\
&\qquad \lambda_{\mathsf{Val}}\,(x,\ \delta,\ t) \to \mathsf{eval}\,(\delta,\ x \mapsto \mathsf{eval}\,\sigma\,u)\,t \\
&\qquad v \qquad\qquad\quad\;\; \to v\,(\mathsf{eval}\,\sigma\,u)
\end{aligned}
$$

$$
\begin{aligned}
&\mathsf{quote} : \mathsf{Val}\,\Gamma \to \mathsf{Tm}\,\Gamma \\
&\mathsf{quote}\,x \qquad\qquad\qquad :\equiv x \\
&\mathsf{quote}\,(\lambda_{\mathsf{Val}}\,(x,\ \delta,\ t)) :\equiv \lambda\,x'.\,\mathsf{quote}\,(\mathsf{eval}\,(\delta,\ x \mapsto x')\,t) \\
&\qquad\qquad\qquad\qquad\qquad\quad \text{where } x' \text{ is fresh in } \Gamma \\
&\mathsf{quote}\,(t\,u) \qquad\qquad\; :\equiv (\mathsf{quote}\,t)\,(\mathsf{quote}\,u)
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In the following we consider Cartesian a CTT with coe, hcom, HITs and Glue.

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Terms are in triple contexts.

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In analogy to MLTT NbE, cubical NbE should take a “semantic interpretation” of the context as input.

- An interval substitution $\sigma : \mathsf{Sub}^{\mathsf{I}}\,\Psi_0\,\Psi_1$.
- A cofibration implication $f : \alpha_0 \Rightarrow \alpha_1[\sigma]$.
- A value environment $\delta : \mathsf{Env}\,\Gamma_0\,(\Gamma_1[\sigma,\ f])$.

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&(f : \alpha_0 \Rightarrow \alpha_1[\sigma]) \\
&(\delta : \mathsf{Env}\, \Gamma_0\, (\Gamma_1[\sigma,\ f])) \\
\to\ &\mathsf{Tm}\,(\Psi_1; \alpha_1; \Gamma_1) \to \mathsf{Val}\,(\Psi_0; \alpha_0; \Gamma_0)
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6 out of 10 inputs are computationally relevant in implementation:

- Ψ_0 marks the next fresh interval variable.

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6 out of 10 inputs are computationally relevant in implementation:

- Ψ_0 marks the next fresh interval variable.
- α_0 is used for “forcing” (see later).
- Γ_0 is passed to detect when there are no fibrant free variables.
- σ, δ and t are evidently required.

Trouble with interval substitution

MLTT NbE: Val substitution is inefficient.

$$-[-] : \mathsf{Val}\,\Delta \to \mathsf{Env}\,\Gamma\,\Delta \to \mathsf{Val}\,\Gamma$$

Evaluation creates **shared structure**. Recursive substitution destroys all such sharing by creating fresh copies of values.

Example for sharing:

$$\mathsf{let}\,x := f\;y\,\mathsf{in}\;(x,\,x,\,x,\,x,\,x)$$

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Likewise: recursive **interval substitution** destroys all structure sharing.

- MLTT NbE: no need for value substitution.
- CTT NbE: must support interval substitution on values.

Cubical NbE

Two extra operations.

1. Interval substitution

$$-[-] : \mathsf{Val}\,(\Psi_0; \alpha; \Gamma) \to (\sigma : \mathsf{Sub}^{\mathsf{I}}\,\Psi_1\,\Psi_0) \to \mathsf{Val}\,(\Psi_1; \alpha[\sigma]; \Gamma[\sigma])$$

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2. Forcing

$$\mathsf{force} : \mathsf{Val}\,(\Psi; \alpha; \Gamma) \to \mathsf{Val}\,(\Psi; \alpha; \Gamma)$$

Computes delayed substitutions sufficiently to yield a *head normal* value.
See also: notion of forcing in lazy evaluation.

Stability annotations

Forcing has trivial cost on canonical values, for example:

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Our implementation:

- Neutrals are annotated with *blocking sets* of interval variables.
- Only an approximation of precise predicates!
- We can quickly see if a substitution has no action on a neutral.

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Forcing w.r.t. cofibrations

Forcing doesn't just compute substitutions, but *cofibration weakening* as well.

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Contrast MLTT NbE: weakening of values has no cost!
(if we use a suitable variable representation in values, e.g. De Bruijn levels)

Closures vs. binders

We can't represent all interval binders with closures!

$$\mathsf{coe}\, r\, r'\, (i.\, A \to B)\, f \equiv \lambda\, x.\, \mathsf{coe}\, r\, r'\, (i.\, B)\, (f\, (\mathsf{coe}\, r'\, r\, (i.\, A)\, x))$$

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- coe, hcom: we need to peek under interval binders, so we use *explicit weakenings* as semantic binders.
- Other cases (e.g. dependent paths, path abstractions): we use closures.

Defunctionalization (1)

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Defunctionalization: representing higher-order functions with first-order data and a first-order generic application.

Defunctionalization (2)

Interval substitution has action on closures:

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(\mathsf{eval}_{\mathsf{cl}}(x,\,\delta,\,t))[\sigma] &\equiv \mathsf{eval}_{\mathsf{cl}}(x,\,\delta[\sigma],\,t)\\
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Seems like a major challenge. In the long term we'd want some *logical framework* for implementing (C)TT evaluation.

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If we don't coerce along Glue, interval substitution only has linear runtime overhead.

Exploiting CTT canonicity (1)

Back to MLTT for a bit:

- Consider closed evaluation of if−then−else.
- The Bool scrutinee is true or false, so we have to evaluate just one of the branches.
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- There are computation rules in *closed evaluation* which evaluate *all* components ("branches") of a system!
- This is bad.

Exploiting CTT canonicity (2)

The offending rules are precisely the hcom rules for strict inductive types.

$$\mathsf{hcom}\, r\, r'\, [\alpha \mapsto i.\, \mathsf{suc}\, t]\, (\mathsf{suc}\, b) \equiv \mathsf{suc}\, (\mathsf{hcom}\, r\, r'\, [\alpha \mapsto i.\, t]\, b)$$

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So we can use this rule instead[2]:

$$\mathsf{hcom}\ r\ r'\ [\alpha \mapsto i.\, t]\ (\mathsf{suc}\ b) \equiv \mathsf{suc}\ (\mathsf{hcom}\ r\ r'\ [\alpha \mapsto i.\, \mathsf{pred}\ t]\ b)$$

pred is a metatheoretic function which unwraps a suc.

---

[2] Used in Simon Huber: *Cubical Interpretations of Type Theory*, sec. 7.2

Exploiting CTT canonicity (3)

The pred rule can be generalized for arbitrary strict inductive types.

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The pred rule can be generalized for arbitrary strict inductive types.

In a purely cubical context (no fibrant variables), no computation rule evaluates all components of a system.

Outline

1. Normalization-by-evaluation for MLTT

2. NbE for CTT

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Implementation

- `https://github.com/AndrasKovacs/cctt`
- It's called `cctt` because it's a Cartesian CTT.
- $\sim$5000 lines of Haskell.
- Features: path types, line types, bidirectional type inference, strict inductive types, parameterized HITs.
- Design is a mixture of AFH, ABCFHL and `cubicaltt`.
  - Systems and ghcom from AFH.
  - Glue type from ABCFHL.
  - HIT implementation from `cubicaltt`.
- No universe checking (type-in-type), no termination checking.
- At least 100 times faster type checking than Agda.

Transporting along Bool negation

Convert Bool negation to a path, compose it with itself N times, transport true over it. Times in seconds.

N	**Agda**	**cctt**	**Ratio**
100	0.29	0.00041	707
250	0.97	0.00095	1021
500	3.36	0.0019	1768
750	7.07	0.0030	2356
1000	12.57	0.0047	2674
10^6	N/A	5.65	N/A

Computing winding numbers

Take an integer, convert it to a path in base $=_{\mathbb{S}^1}$ base, then convert back. Times in seconds.

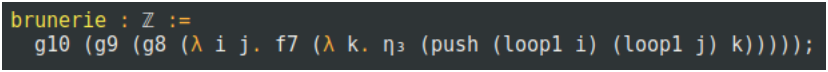
N	**Agda**	**cctt**	**Ratio**
100	0.34	0.0005	680
250	1.89	0.0012	1575
500	5.643	0.0023	2453
750	10.37	0.0043	2411
1000	18.52	0.0059	3138
10^6	N/A	7.98	N/A

Brunerie and the issue with hcom-s (1)

We tried the new Brunerie number definition by Ljungström and Mörtberg[3].

Problem: we did not have ghcom at that point. We had two extra empty hcom-s for each coercion along univalence.

This caused a mismatch with cubical Agda, the following did not typecheck:



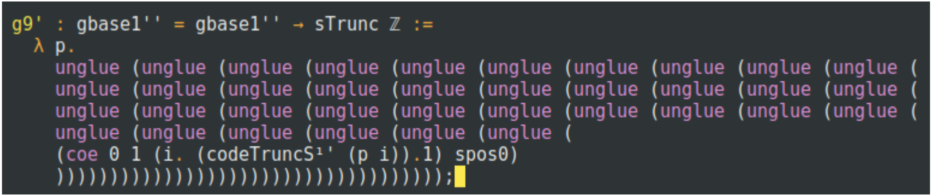
```
brunerie : ℤ :=
  g10 (g9 (g8 (λ i j. f7 (λ k. η₃ (push (loop1 i) (loop1 j) k)))));
```

---

[3] *Formalizing* $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$ *and Computing a Brunerie Number in Cubical Agda*

Brunerie and the issue with hcom-s (2)

Fortunately, I was able to manually insert 18 or 36 Glue types at several places to make it well-typed. One such place:



```
g9' : gbase1'' = gbase1'' → sTrunc ℤ :=
  λ p.
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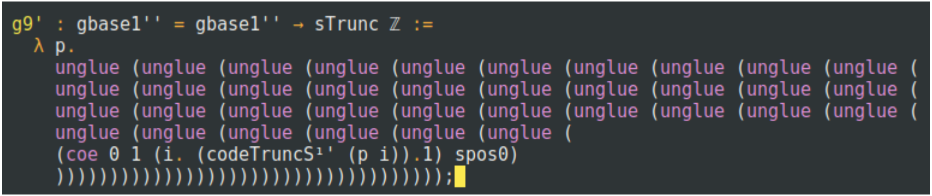

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- Just before the last $g10$ step, we have the set truncation of -2 wrapped in half million empty hcom-s.

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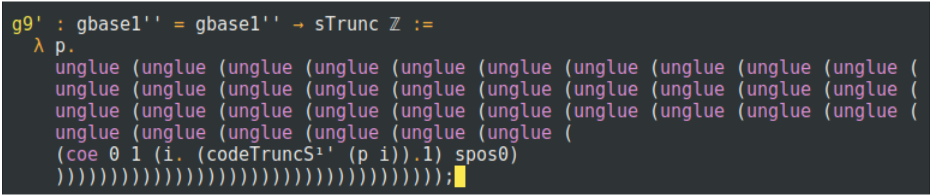

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“Who needs ghcom if we can easily compute a few million empty hcom-s?”

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With the addition of ghcom:

- The Agda-computable Brunerie number definition runs in 0.5 ms, computing a mere 700 hcom-s ($\sim$1 million times speedup!).

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To do:

- Two more variants from Anders & Axel's paper (β_1 and β_2).
- The infamous older `cubicaltt` definitions.

Speedup from De Morgan intervals?

Tom Jack has a $\pi_3(\mathbb{S}^2)$ generator definition:

- Computes instantly in `cubicaltt` (De Morgan CTT).
- Computes in 3 minutes in `cctt`, in 96 million hcom-s.
  (Fun fact: without ghcom, it computes in 20 minutes, in **9.5 billion** hcom-s.)

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The difference *appears to be* the usage of interval connections.

Could we add some connections to Cartesian CTT?

Or: implement a De Morgan CTT with our basic optimizations.

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Can we add this to Agda? Yes. Some things are harder. We’d need a complete rewrite of the Agda Abstract Machine.