Hilton-Milnor's theorem in ∞-topoi.

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Hilton-Milnor's theorem

- First proved by P. Hilton (1955) when $X_i = S^{k_i}$.
- Generalized by J. Milnor (1956) to suspension spaces.

Theorem (HM)

Let $X_1, \cdots, X_n$ be pointed <u>*connected*</u> *spaces. There is a natural homotopy equivalence*

$$\Omega\Sigma(X_1 \vee \cdots \vee X_n) \simeq \prod_{w \in \mathrm{Lie}_n}' \Omega\Sigma w(X_1, \cdots, X_n).$$

1. $\prod'$ denotes a *weak* product.
2. Lie_n is a choice of $\mathbb{Z}$-basis for the *free Lie ring* $\mathrm{Lie}(x_1, \cdots, x_n)$.

Example

$\mathrm{Lie}_2 = \{x, y, [x, y], [x, [x, y]], [y, [x, y]], [x, [x, [x, y]]], \cdots\}$

Motivations and objectives

Goals of this project:

- Formulate and prove the theorem in ∞-categories other than $\mathcal{S}$.
- Prove the theorem *synthetically* (without using any specific model of $(\infty, 1)$-categories).

There has been a lot of work along similar lines !

1. Delooping machine, May's recognition theorem (Lurie)
2. Blackers-Massey theorem (Anel-Biedermann-Finster-Joyal)
3. James' splitting and EHP sequence (Devalapurkar-Haine)

   ...

Plan of the talk

1. Motivational interlude
2. Homotopy theory in ∞-categories
3. Hilton-Milnor's theorem, further directions

Interlude: homotopy operations

Definition

A *homotopy operation* of type $(n_1, \cdots, n_r; m)$ is an assignment

$$\pi_{n_1}(X) \times \cdots \times \pi_{n_1}(X) \longrightarrow \pi_m(X)$$

which is *natural* in the pointed space X.

They are the analogues of Steenrod operations on cohomology !

By Yoneda, these correspond to homotopy classes

$$S^m \longrightarrow S^{n_1} \vee \cdots \vee S^{n_r}$$

ie. to elements in $\pi_m(S^{n_1} \vee \cdots \vee S^{n_r})$.

Interlude: homotopy operations

(HM) tells us that

$$\pi_m(S^{n_1} \vee \cdots \vee S^{n_r}) \cong \bigoplus_{j=i}^{\infty} \pi_m(S^{k_j})$$

for some sequence $j \mapsto k_j$. In fact $k_j = 1 + \sum_i n_i \ell_i(w_j)$.

The bijection goes as follows :

$$(\beta_j)_{j\geq 1} \longmapsto \sum_{j=1}^{\infty} w_j(i_1, \cdots, i_r) \circ \beta_j$$

where $i_i : S^{n_i} \longrightarrow S^{n_1} \vee \cdots \vee S^{n_r}$.

Remark

To apply a word w to the tuple $(i_1, \cdots, i_n)$, we use the Whitehead bracket

$$[-,-] : \pi_{k+1}(X) \otimes \pi_{\ell+1}(X) \longrightarrow \pi_{k+\ell+1}(X).$$

Interlude: homotopy operations

Note that each $\alpha \in \pi_m(S^k)$ gives rise to a (unary) operation of type $(k; m)$

$$\begin{aligned} \pi_k(X) &\longrightarrow \pi_m(X) \\ \beta &\longmapsto \beta \circ \alpha. \end{aligned}$$

Consequence of HM : these unary operations generate them *all* under addition and the Whitehead bracket !

Interlude (continued)

Another exciting consequence of (HM):

For simplicity, take $\Sigma = S^n \vee S^n$ and consider

$$\begin{aligned} \pi_m(S^n) \times \pi_n(\Sigma) &\longrightarrow \pi_m(\Sigma) \\ (\alpha, \beta) &\longmapsto \beta \circ \alpha. \end{aligned}$$

We know this assignment is linear in α. What about the other variable?

Write $i_1, i_2 : S^n \longrightarrow S^n \vee S^n$ for the two inclusions.

(HM) tells us that

$$(i_1 + i_2) \circ \alpha = i_1 \circ \alpha + i_2 \circ \alpha + \sum_{j=3}^{\infty} w_j(i_1, i_2) \circ h_{j-3}(\alpha).$$

The $h_j(\alpha)$ are *higher Hopf invariants* for α.

When $\alpha : S^{2r+1} \longrightarrow S^{r+1}$, one recovers the usual Hopf invariant

$$h_0(\alpha) = H(\alpha)\iota \quad \in \pi_{2r+1}(S^{2r+1}) \cong \mathbb{Z}$$

.

Homotopy theory in ∞-categories

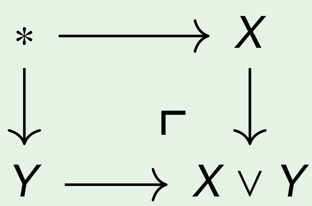
$\mathcal{C} = \infty$-category with finite limits and colimits.

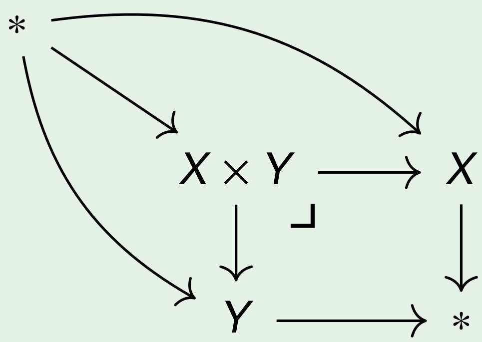
$* =$ a terminal object of $\mathcal{C}$.

$\mathcal{C}_* = \mathcal{C}_{/*} = \infty$-category of pointed objects.

<u>Fact</u>: limits and colimits in $\mathcal{C}_*$ can be computed in $\mathcal{C}$ (appropriately) !

Example





Homotopy theory in ∞-categories

Now we define the elementary operations :

- Suspensions and loop objects

$$
\begin{array}{ccc}
X & \longrightarrow & * \\
\downarrow & \ulcorner & \downarrow \\
* & \longrightarrow & \Sigma X
\end{array}
\qquad\qquad
\begin{array}{ccc}
\Omega X & \longrightarrow & * \\
\downarrow & \lrcorner & \downarrow \\
* & \longrightarrow & X
\end{array}
$$

- Smash and half-smash products

$$
\begin{array}{ccc}
X \vee Y & \longrightarrow & X \times Y \\
\downarrow & \ulcorner & \downarrow \\
* & \longrightarrow & X \wedge Y
\end{array}
\qquad\qquad
\begin{array}{ccc}
Y & \longrightarrow & X \times Y \\
\downarrow & \ulcorner & \downarrow \\
* & \longrightarrow & X \rtimes Y
\end{array}
$$

Homotopy theory in ∞-categories

We also have spheres :

- $S^0 := * \sqcup *$
- $S^n := \Sigma^n S^0 \simeq \operatorname{colim}_{S^n} * =: S^n \otimes *$.

Proposition

- *There is an adjunction* $\Sigma \dashv \Omega \ : \ \mathcal{C}_* \rightleftarrows \mathcal{C}_*$.
- $(\mathcal{C}_*, \wedge, S^0)$ *is a symmetric monoidal* ∞*-category.*
- $\Sigma(X \wedge Y) \simeq \Sigma X \wedge Y \simeq X \wedge \Sigma Y$ *for any* $X, Y \in \mathcal{C}_*$.

Homotopy theory in ∞-categories

Recall: a map $f : X \longrightarrow Y$ is said to be an *effective epimorphism* if its nerve

$$\cdots \;\; X \times_Y X \times_Y X \;\; X \times_Y X \;\; X \xrightarrow{f} Y$$

is a simplicial resolution of Y.

Definition

- A map is (-1)-connected if it is an effective epimorphism.
- When $n \geq 0$, we say that $f : X \longrightarrow Y$ is n-connected if it is an effective epimorphism and $\Delta f : X \longrightarrow X \times_Y X$ is $(n-1)$-connected.

A map $f : X \longrightarrow Y$ is n-connected if and only if its iterated diagonals $\Delta^k f$ are effective epimorphisms for all $0 \leq k \leq n+1$.

A map is ∞-connected if it is n-connected for every $n \geq -2$.

Homotopy theory in ∞-categories

We can also speak of n-connected objects: X is n-connected if $X \to *$ is n-connected.

Now we assume that $\mathcal{E}$ is an ∞-topos.

Proposition

Let $X, Y \in \mathcal{E}_*$ with X k-connected and Y ℓ-connected.

- *ΣX is $(k+1)$-connected.*
- *ΩX is $(k-1)$-connected.*
- *$X \wedge Y$ is $(k+\ell+1)$-connected.*
- *$X^{\wedge n}$ is $(n(k+1)-1)$-connected.*

Proposition

Let $X \xrightarrow{g} Y \xrightarrow{f} Z$ be composable maps in $\mathcal{E}_*$.

- *If f, g are n-connected, then so is fg.*
- *If g is $(n-1)$-connected and fg is n-connected, then f is n-connected.*
- *If fg is n-connected and f is $(n+1)$-connected, then g is n-connected.*

Homotopy theory in ∞-categories

A *group object* in $\mathcal{E}$ is a simplicial object $X : \mathbf{\Delta}^{op} \longrightarrow \mathcal{E}$ such that

- $X_0 \simeq *$
- $X(\Delta^n) \xrightarrow{\sim} X(\Lambda^n_i)$ is an equivalence for *all* n, i.

Here when $K \in$ sSet is a simplicial set, we write

$$X(K) := \int_{[n]\in\mathbf{\Delta}} X_n^{K_n}.$$

Proposition (Splitting lemma)

Let $X \xrightarrow{i} Y \underset{p}{\overset{s}{\leftrightarrows}} Z$ *be a fiber sequence of group objects which is split at the level of underlying pointed objects.*
Then the following composite is an equivalence:

$$X_1 \times Z_1 \xrightarrow{i\times s} Y_1 \times Y_1 \simeq Y_2 \xrightarrow{d_1} Y_1.$$

Hilton-Milnor's theorem (again)

Theorem (L.)

Let $\mathcal{E}$ be an ∞-topos, and $X_1, \cdots, X_n \in \mathcal{E}_*$ connected objects.
There is a natural equivalence

$$\Omega\Sigma(X_1 \vee \cdots \vee X_n) \xleftarrow{\sim} \prod_{w \in \mathrm{Lie}_n}' \Omega\Sigma w(X_1, \cdots, X_n).$$

Outline of the proof:

For simplicity we take $n = 2$ and seek to decompose $\Omega\Sigma(X \vee Y)$.

There is a split fiber sequence in $\mathcal{E}_*$

$$Y \rtimes \Omega X \longrightarrow X \vee Y \rightleftarrows X.$$

Outline of proof

After looping, the splitting lemma yields an equivalence

$$\Omega(X \vee Y) \xleftarrow{\sim} \Omega X \times \Omega(Y \rtimes \Omega X).$$

When X, Y are suspensions, the term $Y \rtimes \Omega X$ splits further and we find

$$\Omega\Sigma(X \vee Y) \xleftarrow{\sim} \Omega\Sigma X \times \Omega\Sigma(Y \vee (Y \wedge \Omega\Sigma X)).$$

Now we use the following result proved in (DH):

Theorem (James' splitting)

If $\mathcal{C}$ is an ∞-category with universal pushouts, and $X \in \mathcal{C}_*$ is a connected object, there is a natural equivalence

$$\Sigma\Omega\Sigma X \xleftarrow{\sim} \bigvee_{i=1}^{\infty} \Sigma X^{\wedge i}.$$

Outline of proof

Combining these results, we find an equivalence

$$J(X \vee Y) \xleftarrow{\sim} JX \times J\left(\bigvee_{i=0}^{\infty} Y \wedge X^{\wedge i}\right)$$

where from now on we write $J = \Omega\Sigma$.

Now iterate this formula :

$$J(X \vee Y) \xleftarrow{\sim} JX \times JR_1 \xleftarrow{\sim} JX \times JY \times JR_2 \xleftarrow{\sim} JX \times JY \times J(X \wedge Y) \times JR_3 \xleftarrow{\sim} \cdots$$

The monomials in X, Y that appear are exactly those $w(X, Y)$ with $w \in \mathrm{Lie}_2$!

Outline of proof

Iterating, we obtain a tower

$$\begin{array}{ccccccccccc}
* & \longrightarrow & JX & \longrightarrow & \cdots & \longrightarrow & \prod_i JX_i & \longrightarrow & \cdots & & \prod_i' JX_i \\
\downarrow & & \downarrow & & & & \downarrow & & & & \Big\downarrow h_{X,Y} \\
J(X \vee Y) & \xleftarrow{\sim} & JX \times JR_1 & \xleftarrow{\sim} & \cdots & \xleftarrow{\sim} & \prod_i JX_i \times JR_n & \xleftarrow{\sim} & \cdots & & \\
\| & & \wr\downarrow & & & & \wr\downarrow & & & & \\
JR_0 & = & JR_0 & = & \cdots & = & JR_0 & = & \cdots & & JR_0
\end{array}$$

where $R_0 = X \vee Y$ and $X_i = w_i(X, Y)$.

Now we use connectivity estimates.

The connectivity of $Jw_i(X, Y)$ and JR_i tend to ∞ with i.

Outline of proof

The stability properties of n-connected maps imply that $h_{X,Y}$ is ∞-connected. This works in *any* ∞-topos!

Now we use the following trick :

since $\mathcal{E}$ is an ∞-topos, we can chose a presentation

$$\mathcal{E} \underset{}{\overset{L}{\leftrightarrows}} \mathcal{P}(\mathcal{C})$$

where L is a left exact left adjoint.

By tracking down the explicit construction of $h_{X,Y}$, we see that

$$Lh_{X,Y} = h_{LX,LY}.$$

But $\mathcal{P}(\mathcal{C})$ is hypercomplete, so $h_{LX,LY}$ is an equivalence !

Further directions

- Can we translate this proof in HoTT ? The trick doesn't work anymore, so one might have to postulate Whitehead's principle.
- When $\mathcal{E}$ is an ∞-topos, there is an equivalence

$$\mathrm{Grp}(\mathcal{E}) \underset{B}{\overset{\Omega}{\rightleftarrows}} \mathcal{E}_*^{\geq 1}$$

  through which $\Omega\Sigma$ becomes the *free group functor*. In this context, (HM) becomes a theorem about free groups, which suggests an extension of the notion n-nilpotence defined by Biedermann-Dwyer.
- Is Lie_n really just an indexing set ? How can one perform the Magnus construction in a homotopy coherent setting ?

The End

Thank you for listening !

References

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