Symmetric Monoidal Smash Products in HoTT

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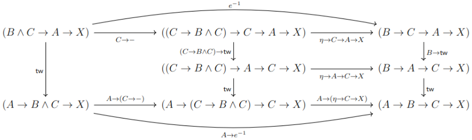
Introduction

- The smash product plays a crucial role in homotopy theory
- Key property: it is (1-coherent) **symmetric monoidal**
- This fact is useful when doing HoTT too:
  - Brunerie (2016): $\pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$
  - Van Doorn (2018): Cohomological spectral sequences
- Problem: this fact has never been fully proved in HoTT

A brief history of the smash product in HoTT

The pretty approach

- Van Doorn (2018) *almost* proved the theorem
- Used an argument from *closed monoidal categories*
- Only lacked one tiny technical lemma



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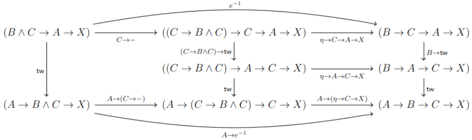
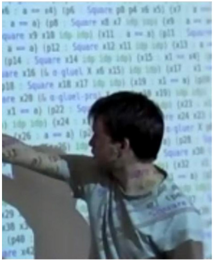


The madman approach

- Brunerie (2018) wrote an Agda program generating Agda code for the proof
- Problem: Agda couldn't type-check all proofs without running out of memory



Another approach

- Today we present a new approach
- The goal: make smash products in HoTT less scary by introducing a new heuristic
- This heuristic can be used (with some manual labour) to show the theorem at hand.
- Somewhat more involved proofs than van Doorn's but definitely shorter than Brunerie's .agda-file.

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- a coherence $\mathsf{push_{lr}} : \mathsf{push_l}\, \star_A = \mathsf{push_r}\, \star_B$

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$$\begin{array}{ccc} A \vee B & \longrightarrow & A \times B \\ \big\downarrow & & \big\downarrow \\ 1 & \longrightarrow & A \wedge B \end{array}$$

The pentagon

Fact

The smash product is associative. We use $\alpha_{A,B,C} : (A \wedge B) \wedge C \xrightarrow{\sim} A \wedge (B \wedge C)$ to denote the associator.

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The 'impossible' pentagon axiom for $\wedge$:

$$\begin{array}{ccccc}
 & & ((A \wedge B) \wedge C) \wedge D & & \\
 & \swarrow^{\alpha_{A,B,C} \wedge 1_D} & & \searrow^{\alpha_{A \wedge B,C,D}} & \\
(A \wedge (B \wedge C)) \wedge D & & & & (A \wedge B) \wedge (C \wedge D) \\
\downarrow{\scriptstyle \alpha_{A,B \wedge C,D}} & & & & \downarrow{\scriptstyle \alpha_{A,B,C \wedge D}} \\
A \wedge ((B \wedge C) \wedge D) & & \xrightarrow{1_A \wedge \alpha_{B,C,D}} & & A \wedge (B \wedge (C \wedge D))
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By *pentagonator*, I will mean the function described by either side of the pentagon.

The pentagon

- Why is it so hard to verify?
- Proving it amounts to constructing a homotopy

$$(x : ((A \wedge B) \wedge C) \wedge D) \to f\, x = g\, x$$

for the pentagonators f and g.

Induction hell

```
pf : (x : ((A ∧· B) ∧· C) ∧ D)  → f x ≡ g x
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A first step

- **Need:** a better way to deal with equalities of functions $f : \bigwedge_i A_i \to B$

Lemma 2

To check $f = g$ for $f, g : A \wedge B \to C$, the coherence for $\mathsf{push}_{\mathsf{lr}}$ *is automatic.*

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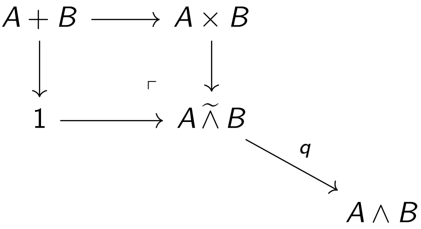

We have $(f \circ q = g \circ q) \implies (f = g)$

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- Still: 22 (highly non-trivial) cases left...

Interlude: homogeneous types

Definition 3

A pointed type A is homogeneous if for every $a : A$, there is an automorphism $e_a : A \simeq A$ such that $e_a \star_A = a$

- All (pointed) path spaces are homogeneous.

Lemma 4 (Evan's Trick)

*Let $f, g : A \to_\star B$ be two pointed functions with B homogeneous. If there is a homotopy $(x : A) \to f\,x = g\,x$, then $f = g$ **as pointed functions**.*

Interlude: homogeneous types

Lemma 5 (Evans's trick 2.0)

Let $f, g : A \wedge B \to_\star C$ be two pointed functions with C homogeneous. If there is a homotopy

$$((x, y) : A \times B) \to f\langle x, y\rangle = g\langle x, y\rangle$$

then $f = g$ (as pointed functions)

Proof.

Using the adjunction $(A \wedge B \to_\star C) \simeq A \to_\star (B \to_\star C)$. □

- Dream: Apply the trick to pentagon.

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- ~~Dream: Apply the trick to pentagon.~~
- Nightmare: We can't (the codomain is **not** homogeneous).

The heuristic

- Fortunately, there is still hope: loop spaces are homogeneous. Let's 'make them appear' in the proof of the pentagon.

Definition 6

Let $f, g : A \wedge B \to_\star C$. A homotopy $h : ((a, b) : A \times B) \to f\langle a, b\rangle = g\langle a, b\rangle$ induces two functions

- $L_h : A \to \Omega C$
- $R_h : B \to \Omega C$

- For instance, $L_h\, a$ is defined by the composition

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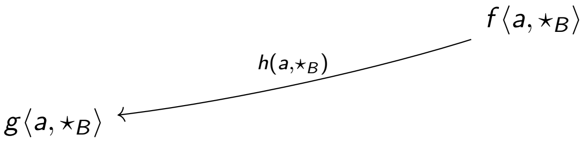

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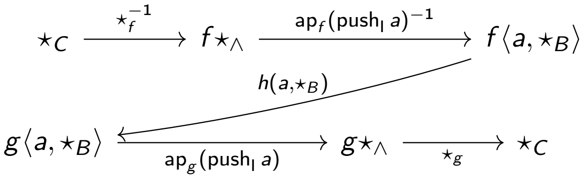

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Lemma 7

If $L_h = \text{const}_{(L_h \star_A)}$ *and* $R_h = \text{const}_{(R_h \star_B)}$, *then* $f = g$

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- The point: applying this construction to the pentagonators $f, g : ((A \wedge B) \wedge C) \wedge D \to A \wedge (B \wedge (C \wedge D))$, the function L_h is of type

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Homogeneous codomain!

The heuristic

- We want to prove that L_h is constant. This is precisely where the explosion of complexity happens in a naive proof...
- ...but thanks to our set up: enough to show that

$$A \times B \times C \xrightarrow{\langle -,-,- \rangle} (A \wedge B) \wedge C \xrightarrow{L_h} \Omega(A \wedge (B \wedge (C \wedge D)))$$

  is constant.
- Amounts to checking the actions of f and g on $\mathsf{push}_{\mathsf{l}}\langle a, b, c\rangle$, but no further coherences!
  - In particular: no nestled $\mathsf{push}_{\mathsf{l}}$ and $\mathsf{push}_{\mathsf{r}}$ constructors.
  - Only ~~13 cases~~ 1 case to check

The heuristic

- By iterating the argument, we may use L_h and R_h to construct equalities $f = g$ for any $f, g : \bigwedge_{i \leq n} A_i \to B$.
- **Heuristic**: We only need to construct a homotopy $h : f\langle x_1, \ldots, x_n\rangle = g\langle x_1, \ldots, x_n\rangle$ and show that it is compatible with ap_f and ap_g on *single* applications of push_l and push_r.
- Number of cases: ~~$O(2^n)$~~ $O(2n)$

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- Number of cases: ~~$O(2^n)$~~ $O(2n)$
- For instance: for the pentagonators, we only need to provide 7 pieces of (low-dimensional) data (instead of 29).

Lemma 6. *For any two functions $f, g : ((A \wedge B) \wedge C) \wedge D) \to E$, the following data gives an equality $f = g$:*

(i) A homotopy $h : ((a, b, c, d) : A \times B \times C \times D) \to f\langle a, b, c, d\rangle = g\langle a, b, c, d\rangle$.

(ii) For every triple $(a, b, d) : A \times B \times D$, a filler of the square

$$\begin{array}{ccc}
f\langle \star_A, \star_B, \star_C, d\rangle & \xrightarrow{h(\star_A,\star_B,\star_C)} & g\langle \star_A, \star_B, \star_C, d\rangle \\
{\scriptstyle \mathsf{ap}_{f\langle -,\star_C,d\rangle}(\mathsf{push_r}(\star_B)^{-1})}\uparrow & & \uparrow{\scriptstyle \mathsf{ap}_{g\langle -,\star_C,d\rangle}(\mathsf{push_r}(\star_B)^{-1})} \\
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f\langle a, b, \star_C, d\rangle & \xrightarrow[h(a,b,\star_C)]{} & g\langle a, b, \star_C, d\rangle
\end{array}$$

(iii) For every pair $(c, d) : C \times D$, a filler of the square

$$
\begin{array}{ccc}
f\langle \star_A, \star_B, \star_C, d\rangle & \xrightarrow{h(\star_A,\star_B,\star_C,d)} & g\langle \star_A, \star_B, \star_C\rangle \\
\mathsf{ap}_{f\langle -,\star_C,d\rangle}(\mathsf{push_r}(\star_B))^{-1} \Big\uparrow & & \Big\uparrow \mathsf{ap}_{g\langle -,\star_C,d\rangle}(\mathsf{push_r}(\star_B))^{-1} \\
f\langle \star_\wedge, \star_C, d\rangle & & g\langle \star_\wedge, \star_C\rangle \\
\mathsf{ap}_{f\langle -,d\rangle}(\mathsf{push_l}(\star_\wedge))^{-1} \Big\uparrow & & \Big\uparrow \mathsf{ap}_{g\langle -,d\rangle}(\mathsf{push_l}(\star_\wedge))^{-1} \\
f\langle \star_\wedge, d\rangle & & g\langle \star_\wedge, d\rangle \\
\mathsf{ap}_{f\langle -,d\rangle}(\mathsf{push_r}(c)) \Big\uparrow & & \Big\uparrow \mathsf{ap}_{g\langle -,d\rangle}(\mathsf{push_r}(c)) \\
f\langle \star_\wedge, c, d\rangle & & g\langle \star_\wedge, c, d\rangle \\
\mathsf{ap}_{f\langle -,c,d\rangle}(\mathsf{push_r}(\star_B)) \Big\uparrow & & \Big\uparrow \mathsf{ap}_{g\langle -,c,d\rangle}(\mathsf{push_r}(\star_B)) \\
f\langle \star_A, \star_B, c, d\rangle & \xrightarrow[h(\star_A,\star_B,c,d)]{} & g\langle \star_A, \star_B, c, d\rangle
\end{array}
$$

(iv) For every triple $(a, c, d) : A \times C \times D$, a filler of the square

$$
\begin{array}{ccc}
f\langle \star_A, \star_B, c, d\rangle & \xrightarrow{h(\star_A, \star_B, c, d)} & g\langle \star_A, \star_B, c, d\rangle \\
{\scriptstyle \mathsf{ap}_{f\langle -, c, d\rangle}(\mathsf{push_l}(\star_A))^{-1}} \uparrow & & \uparrow {\scriptstyle \mathsf{ap}_{g\langle -, c, d\rangle}(\mathsf{push_l}(\star_A))^{-1}} \\
f\langle \star_\wedge, c, d\rangle & & g\langle \star_\wedge, c, d\rangle \\
{\scriptstyle \mathsf{ap}_{f\langle -, c, d\rangle}(\mathsf{push_l}(a))} \uparrow & & \uparrow {\scriptstyle \mathsf{ap}_{g\langle -, c, d\rangle}(\mathsf{push_l}(a))} \\
f\langle a, \star_B, c, d\rangle & \xrightarrow[h(a, \star_B, c, d)]{} & g\langle a, \star_B, c, d\rangle
\end{array}
$$

(v) For every triple $(b, c, d) : B \times C \times D$*, a filler of the square*

$$
\begin{array}{ccc}
f\langle \star_A, \star_B, c, d\rangle & \xrightarrow{h(\star_A, \star_B, c, d)} & g\langle \star_A, \star_B, c, d\rangle \\
{\scriptstyle \mathsf{ap}_{f\langle -,c,d\rangle}(\mathsf{push_r}(\star_B))^{-1}} \uparrow & & \uparrow {\scriptstyle \mathsf{ap}_{g\langle -,c,d\rangle}(\mathsf{push_r}(\star_B))^{-1}} \\
f\langle \star_\wedge, c, d\rangle & & g\langle \star_\wedge, c, d\rangle \\
{\scriptstyle \mathsf{ap}_{f\langle -,c,d\rangle}(\mathsf{push_r}(b))} \uparrow & & \uparrow {\scriptstyle \mathsf{ap}_{g\langle -,c,d\rangle}(\mathsf{push_r}(b))} \\
f\langle \star_A, b, c, d\rangle & \xrightarrow[h(\star_A, b, c, d)]{} & g\langle \star_A, b, c, d\rangle
\end{array}
$$

(vi) For every triple $(a, b, c) : A \times B \times C$, a filler of the square

$$
\begin{array}{ccc}
f\langle \star_A, \star_B, \star_C \star_D\rangle & \xrightarrow{h(\star_A, \star_B, \star_C \star_D)} & g\langle \star_A, \star_B, \star_C \star_D\rangle \\
{\scriptstyle \mathsf{ap}_{f\langle -, \star_C, \star_D\rangle}(\mathsf{push_l}(\star_A)))^{-1}} \uparrow & & \uparrow {\scriptstyle \mathsf{ap}_{g\langle -, \star_C, \star_D\rangle}(\mathsf{push_l}(\star_A)))^{-1}} \\
f\langle \star_\wedge, \star_C, \star_D\rangle & & g\langle \star_\wedge, \star_C, \star_D\rangle \\
{\scriptstyle \mathsf{ap}_{f\langle -, \star_D\rangle}(\mathsf{push_l}(\star_\wedge)))^{-1}} \uparrow & & \uparrow {\scriptstyle \mathsf{ap}_{g\langle -, \star_D\rangle}(\mathsf{push_l}(\star_\wedge)))^{-1}} \\
f\langle \star_\wedge, \star_D\rangle & & g\langle \star_\wedge, \star_D\rangle \\
{\scriptstyle \mathsf{ap}_f(\mathsf{push_l}(\star_\wedge)))^{-1}} \uparrow & & \uparrow {\scriptstyle \mathsf{ap}_g(\mathsf{push_l}(\star_\wedge)))^{-1}} \\
f(\star_\wedge) & & g(\star_\wedge) \\
{\scriptstyle \mathsf{ap}_f(\mathsf{push_l}\langle a,b,c\rangle)} \uparrow & & \uparrow {\scriptstyle \mathsf{ap}_g(\mathsf{push_l}\langle a,b,c\rangle)} \\
f\langle a, b, c, \star_D\rangle & \xrightarrow[h(a,b,c,\star_D)]{} & g\langle a, b, c, \star_D\rangle
\end{array}
$$

(vii) For every $d : D$, a filler of the square

$$
\begin{array}{ccc}
f\langle \star_A, \star_B, \star_C, \star_D\rangle & \xrightarrow{h(\star_A,\star_B,\star_C,\star_D)} & g\langle \star_A, \star_B, \star_C, \star_D\rangle \\
\mathsf{ap}_{f\langle -,\star_C,\star_D\rangle}(\mathsf{push_l}(\star_A))^{-1}\Big\uparrow & & \Big\uparrow \mathsf{ap}_{g\langle -,\star_C,\star_D\rangle}(\mathsf{push_l}(\star_A))^{-1} \\
f\langle \star_\wedge, \star_C, \star_D\rangle & & g\langle \star_\wedge, \star_C, \star_D\rangle \\
\mathsf{ap}_{f\langle -,\star_D\rangle}(\mathsf{push_l}(\star_\wedge))^{-1}\Big\uparrow & & \Big\uparrow \mathsf{ap}_{g\langle -,\star_D\rangle}(\mathsf{push_l}(\star_\wedge))^{-1} \\
f\langle \star_\wedge, \star_D\rangle & & g\langle \star_\wedge, \star_D\rangle \\
\mathsf{ap}_{f}(\mathsf{push_l}(\star_\wedge))^{-1}\Big\uparrow & & \Big\uparrow \mathsf{ap}_{g}(\mathsf{push_l}(\star_\wedge))^{-1} \\
f(\star_\wedge) & & g(\star_\wedge) \\
\mathsf{ap}_{f}(\mathsf{push_r}(d))\Big\uparrow & & \Big\uparrow \mathsf{ap}_{g}(\mathsf{push_r}(d)) \\
f\langle \star_\wedge, d\rangle & & g\langle \star_\wedge, d\rangle \\
\mathsf{ap}_{f\langle -,d\rangle}(\mathsf{push_l}(\star_\wedge))\Big\uparrow & & \Big\uparrow \mathsf{ap}_{g\langle -,d\rangle}(\mathsf{push_l}(\star_\wedge)) \\
f\langle \star_\wedge, \star_C, d\rangle & & g\langle \star_\wedge, \star_C, d\rangle \\
\mathsf{ap}_{f\langle -,\star_C,d\rangle}(\mathsf{push_l}(\star_A))\Big\uparrow & & \Big\uparrow \mathsf{ap}_{g\langle -,\star_C,d\rangle}(\mathsf{push_l}(\star_A)) \\
f\langle \star_A, \star_B, \star_C, d\rangle & \xrightarrow[h(\star_A,\star_B,\star_C,d)]{} & g\langle \star_A, \star_B, \star_C, d\rangle
\end{array}
$$

Reaping the fruits

Theorem 8

The smash product satisfies the pentagon identity.

Proof.

After applying of the heuristic, the remaining coherences are easily verified by hand. □

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Theorem 9

The smash product is symmetric monoidal with the booleans as unit.

Summary

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  2. A proof of the symmetric monoidal nature of the smash product

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Thanks for listening!

$$f = g$$

$$\downarrow \qquad\qquad \searrow$$

$$h : \begin{cases} ((\bar{x}, x_n) : (\bigwedge_{i<n} A_i) \times A_n) \\ \to f\langle \bar{x}, x_n \rangle = g\langle \bar{x}, x_n \rangle \end{cases} \qquad \begin{cases} L_h\langle x_1, \ldots\ x_{n-1} \rangle = \mathsf{const} \\ R_h\, x_n = \mathsf{const} \end{cases}$$

$$\downarrow \qquad\qquad \searrow$$

$$h_n : \begin{cases} ((\bar{x}, x_{n-1}) : (\bigwedge_{i<n-1} A_i) \times A_{n-1}) \\ \to f\langle \bar{x}, x_{n-1}, x_n \rangle = g\langle \bar{x}, x_{n-1}, x_n \rangle \end{cases} \qquad \begin{cases} L_{h_n}\langle x_1, \ldots\ x_{n-2} \rangle = \mathsf{const} \\ R_{h_n}\, x_{n-1} = \mathsf{const} \end{cases}$$

$$\downarrow \qquad\qquad \searrow$$

$$\vdots \qquad\qquad \vdots$$

$$\downarrow$$

$$f\langle x_1, \ldots, x_n \rangle = g\langle x_1, \ldots, x_n \rangle$$