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INTERNAL HIGHER TOPOS THEORY

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Internal vs parametrised mathematics

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$\mathcal{B}$-categories

Definition ($\mathcal{B}$-categories and $\mathcal{B}$-groupoids)

A $\mathcal{B}$-category C is a complete Segal object in $\mathcal{B}$, i.e. a functor $\mathsf{C}\colon \Delta^{\mathrm{op}} \to \mathcal{B}$ satisfying the Segal condition and univalence. A $\mathcal{B}$-groupoid is a constant simplicial object in $\mathcal{B}$
$\rightsquigarrow \mathcal{B} \simeq \mathrm{Grpd}(\mathcal{B}) \subset \mathrm{Cat}(\mathcal{B}) \subset \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{B})$

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Proposition ($\mathcal{B}$-categories are sheaves of ∞-categories)

There is an equivalence of ∞-categories $\mathrm{Cat}(\mathcal{B}) \simeq \mathrm{Fun}^{\lim}(\mathcal{B}^{\mathrm{op}}, \mathrm{Cat}_{\infty})$ between the ∞-category of $\mathcal{B}$-categories and the ∞-categories of *sheaves* of ∞-categories on $\mathcal{B}$.

Categorical structure of $\mathrm{Cat}(\mathcal{B})$

Presentability $\mathrm{Cat}(\mathcal{B})$ is presentable $\rightsquigarrow$ has all limits and colimits

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Global sections The geometric morphism $\mathrm{const}\colon \mathcal{S} \leftrightarrows \mathcal{B} :\Gamma$ induces an adjunction

$$\mathrm{const}\colon \ \mathrm{Cat}_\infty \leftrightarrows \mathrm{Cat}(\mathcal{B}) \ :\Gamma$$

$\rightsquigarrow$ can regard ∞-categories as *constant* $\mathcal{B}$-categories.
$\rightsquigarrow$ every $\mathcal{B}$-category has an underlying ∞-category of global sections.

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$(\infty, 2)$-categorical structure $\mathrm{Cat}(\mathcal{B})$ is Cat_∞-enriched via

$\mathrm{Fun}_{\mathcal{B}}(-,-) = \Gamma \underline{\mathsf{Fun}}_{\mathcal{B}}(-,-)$
$\rightsquigarrow$ can be regarded as an $(\infty, 2)$-category
$\rightsquigarrow$ has an intrinsic notion of adjunctions

Internal limits and colimits

Definition (internal limits and colimits)

I, C $\mathcal{B}$-categories $\rightsquigarrow \lim_{\mathsf{I}} \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C}) \to \mathsf{C}$ and $\operatorname{colim}_{\mathsf{I}} \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C}) \to \mathsf{C}$ are the right and left adjoint of the diagonal $\operatorname{diag}_{\mathsf{I}} \colon \mathsf{C} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})$.

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Example: colimits indexed by ∞-categories

$\mathcal{I}$ ∞-category $\rightsquigarrow$ $\operatorname{colim}_{\operatorname{const}(\mathcal{I})} \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\operatorname{const}(\mathcal{I}), \mathsf{C}) \to \mathsf{C}$ recovers $\operatorname{colim}_{\mathcal{I}} \colon \operatorname{Fun}(\mathcal{I}, \Gamma(\mathsf{C})) \to \Gamma(\mathsf{C})$ on global sections.

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Example: colimits indexed by $\mathcal{B}$-groupoids

$A \in \mathcal{B}$ any object $\rightsquigarrow \operatorname{colim}_{A} \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(A, \mathsf{C}) \to \mathsf{C}$ recovers the left adjoint $\pi_! \colon \mathsf{C}(A) \to \mathsf{C}(1) = \Gamma(\mathsf{C})$ of $\pi^* \colon \mathsf{C}(1) \to \mathsf{C}(A)$ on global sections.

Internal presheaves

The universe $\mathcal{B}$ has descent $\leftrightsquigarrow A \mapsto \mathcal{B}_{/A}$ defines a sheaf of ∞-categories on $\mathcal{B}$
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Presheaf $\mathcal{B}$-categories Can now define $\underline{\mathsf{PSh}}(\mathsf{C}) = \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \Omega)$ for every $\mathcal{B}$-category C.

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Theorem (Universal property of internal presheaves)

There is a fully faithful functor $\mathsf{C} \hookrightarrow \underline{\mathrm{PSh}}(\mathsf{C})$ (the Yoneda embedding) that exhibits $\underline{\mathrm{PSh}}(\mathsf{C})$ as the free cocompletion of C.

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In particular, the universe Ω is freely generated by the point under internal colimits.

$\mathcal{B}$-topoi

Definition ($\mathcal{B}$-topos)

A $\mathcal{B}$-category X is a $\mathcal{B}$-topos if it arises as a left exact and accessible localisation

$$\begin{tikzcd} \mathsf{X} \arrow[rr, hook, bend right] & \perp & \underline{\mathsf{PSh}}(\mathsf{C}) \arrow[ll, bend right] \end{tikzcd}$$

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Fully faithfulness $\mathsf{X}_1 \to \underline{\mathsf{PSh}}(\mathsf{C})_1 \times_{\underline{\mathsf{PSh}}(\mathsf{C})_0 \times \underline{\mathsf{PSh}}(\mathsf{C})_0} (\mathsf{X}_0 \times \mathsf{X}_0)$ is an equivalence.

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Accessibility $\mathsf{X} \hookrightarrow \underline{\mathsf{PSh}}(\mathsf{C})$ commutes with certain internally filtered colimits.

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Example (the initial $\mathcal{B}$-topos)

The universe Ω is the *initial* $\mathcal{B}$-topos: for every $\mathcal{B}$-topos X, there is a unique cocontinuous and left exact functor $f^*\colon \Omega \to \mathsf{X}$.

Descent

The $\mathcal{B}$-category of $\mathcal{B}$-categories $A \mapsto \mathrm{Cat}(\mathcal{B}_{/A})$ preserves limits
$\rightsquigarrow$ obtain the $\mathcal{B}$-category $\mathsf{Cat}_{\mathcal{B}} = \mathrm{Cat}(\mathcal{B}_{/-})$ of $\mathcal{B}$-categories.

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Theorem (Chacterisation of $\mathcal{B}$-topoi via descent)

A $\mathcal{B}$-category X is a $\mathcal{B}$-topos if and only if X is presentable and has descent.

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Theorem ($\mathcal{B}$-topoi are equivalent to ∞-topoi over $\mathcal{B}$)

The datum of a $\mathcal{B}$-topos X is equivalent to that of a geometric morphism of ∞-topoi $\mathcal{X} \to \mathcal{B}$.

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$\rightsquigarrow$ need only show that $\mathsf{X} \simeq \mathsf{X}(1)_{/f^*(-)}$ for every $\mathcal{B}$-topos X.

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\begin{tikzcd}
1_{\mathsf{X}(A)} \arrow[d, mapsto] \arrow[rrr, mapsto, bend left] & & & \\
\pi_!(1_{\mathsf{X}(A)}) \arrow[r, mapsto] & \mathsf{X}(1)_{/\pi_!(1_{\mathsf{X}(A)})} & \simeq & \mathsf{X}(A)
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A \arrow[r, mapsto] & f^*(A) \simeq \pi_!(1_{\mathsf{X}(A)}) \\
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This can be made functorial in A, so that one obtains

$$\mathsf{X} \simeq \mathsf{X}(1)_{/f^*(-)}.$$

$\square$

QUESTIONS?