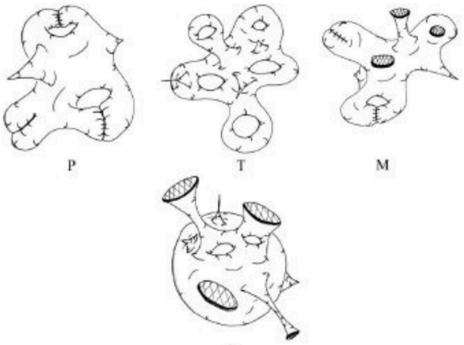
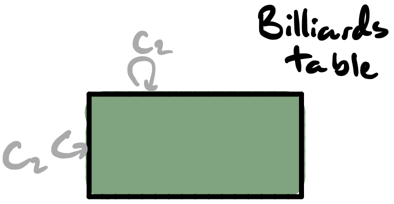
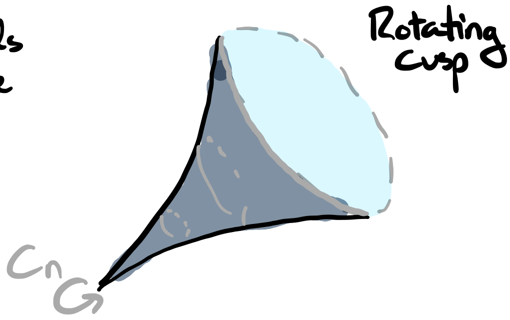
Synthetic Orbifolds
in
Cohesive Homotopy Type Theory

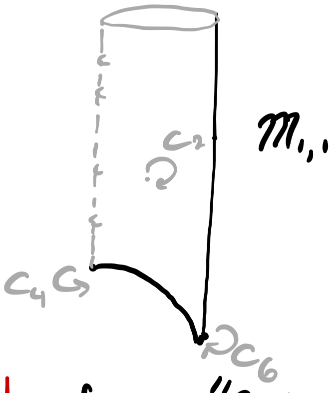


David Jaz Myers

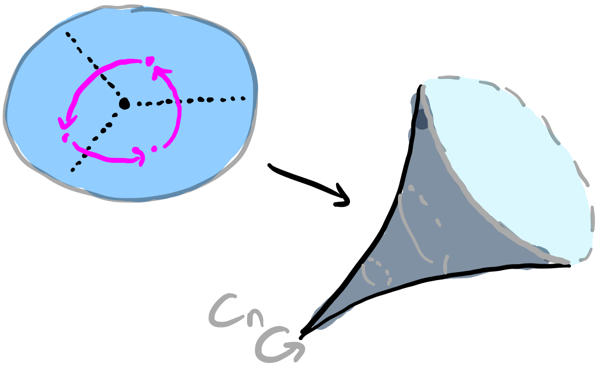
Orbifolds are "smooth spaces" where the points have finite symmetries.

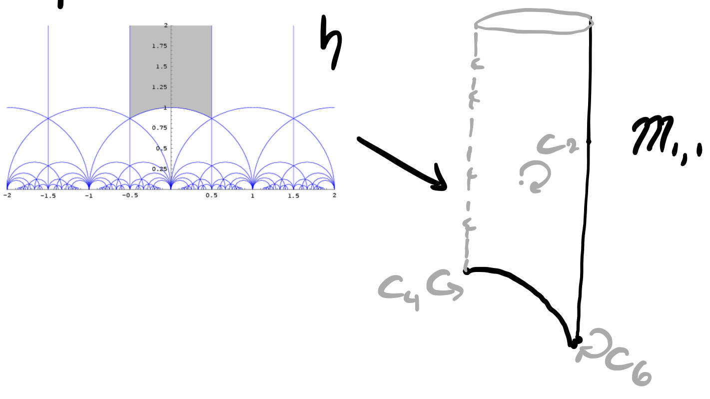






A **good** orbifold is the **homotopy quotient** of a "smooth space" by the action of a discrete group.





Orbifolds, classically:

"weak quotient"

If $\Gamma \circlearrowright X$ proper discontinuously, $X/\!/\Gamma$ is a "good" orbifold.

$X/\!/\Gamma$ is presented by the action groupoid

$$\begin{tikzcd}
(X/\!/\Gamma)_1 \arrow[d, shift right=1.5ex, "s"'] \arrow[d, shift left=1.5ex, "t"] & := & \{(x,y,\gamma) \mid \gamma\cdot x = y\} \arrow[d, shift right=1.5ex, "\mathrm{fst}"'] \arrow[d, shift left=1.5ex, "\mathrm{snd}"] \\
(X/\!/\Gamma)_0 \arrow[u] & := & X \arrow[u]
\end{tikzcd}$$

Special features:

1) $s : (X/\!/\Gamma)_1 \to (X/\!/\Gamma)_0$ is étale.
2) $(s,t) : (X/\!/\Gamma)_1 \to (X/\!/\Gamma)_0^2$ is proper.

} $X/\!/\Gamma$ is proper étale.

Thm (Moerdijk-Pronk):

All orbifolds are presented by proper étale groupoids.

Orbifolds, synthetically:

Orbifolds are "smooth spaces" where the points have finite symmetries.

$\int \dashv \flat \dashv \sharp$ $\qquad$ $(-)^D \dashv (-)^{1/D}$

Working in Cohesive HoTT & Synthetic Differential Geometry:

↳ crisp types internalize the external

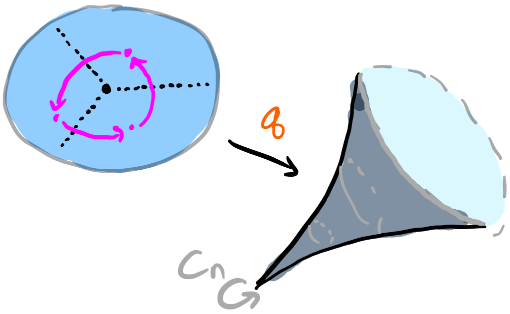
infinitesimals give synthetic calculus

Def: An orbifold is a microlinear type whose types of identifications are properly finite. ← discrete subquotients of finite sets.

Thm: The Rezk completion of a crisp*, ordinary*, proper étale pregroupoid* is an orbifold.

* all ways of saying "the usual, external, proper étale groupoids"

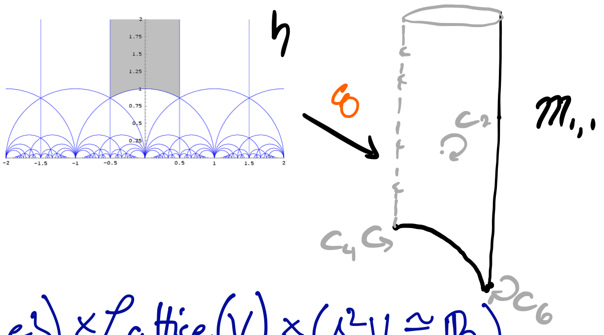
Examples of Synthetic Orbifolds:



① $\mathbb{C} /\!/ \mu_n := (V : \{\text{1-dim } \mathbb{C}\text{-vector space}\})$
$\times\, (T \subseteq V, \text{ a } \mu_n\text{-torsor})$
$\times\, V$

$q(z) := (\mathbb{C}, \mu_n, z)$

② $\mathcal{M}_{1,1} := (\omega : \{\text{1-dim } \mathbb{C}\text{-vector space}\})$
$\times \{\text{Lattice in } V\}$



$q(\tau) := (\mathbb{C}, \mathbb{Z} \oplus \tau\mathbb{Z})$

Define $BSL_2(\mathbb{Z}) := (V : \{\text{2-dim } \mathbb{R} \text{ vectorspace}\}) \times \mathcal{L}attice(V) \times (\Lambda^2 V \cong \mathbb{R})$
$s : \mathcal{M}_{1,1} \to BSL_2(\mathbb{Z}) := (\omega, \ell) \mapsto (\omega, \ell, \text{“}(v,w) \mapsto -\mathrm{im}(v\bar{w})\text{”})$

Thm: s is the $\int$-unit of $\mathcal{M}_{1,1}$.

Def: If $\Gamma : BSL_2(\mathbb{Z}) \to \mathrm{FiniteSet}$ is some finite structure of lattices and $\kappa : \mathbb{N}$, then a modular form of level Γ and weight κ is
$f : ((\omega, \ell) : \mathcal{M}_{1,1})(\gamma : \Gamma(s(\omega, \ell))) \to \omega^{\otimes \kappa}$ which is holomorphic and bounded on 𝔥

SDG, really quickly:

A field $\mathbb{R}$, the smooth reals.

Def (Penon): A number $x : \mathbb{R}$ is infinitesimal if it is not distinct from 0: $\neg\neg(x = 0)$.] $D :\equiv \{x \mid \neg\neg(x=0)\}$.

Axioms: (Some of them)

- (Kock-Lawvere) Any function $f(\varepsilon)$ of a number $\varepsilon^2 = 0$ is linear.
- D is tiny: $X \mapsto X^D$ has an external right adjoint.

Def (Bergeron): A type X is microlinear if for any square

$$\begin{tikzcd} V_1 \arrow[r] \arrow[d] & V_3 \arrow[d] \\ V_2 \arrow[r] & V_4 \end{tikzcd}$$

such that

$$\begin{tikzcd} \mathbb{R}^{V_4} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{R}^{V_3} \arrow[d] \\ \mathbb{R}^{V_2} \arrow[r] & \mathbb{R}^{V_1} \end{tikzcd}$$

, then

$$\begin{tikzcd} X^{V_4} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^{V_3} \arrow[d] \\ X^{V_2} \arrow[r] & X^{V_1} \end{tikzcd}$$

.

of infinitesimal varieties

... includes all manifolds.

Tiny Types

Def: A crisp type T is tiny when:

1) For crisp X, there is $X^{1/T}$ and $\xi : (X^{1/T})^T \to X$.

2) The map

$$\omega \mapsto v \mapsto \xi(\omega \circ v) : (X \to Y^{1/T}) \longrightarrow (X^T \to Y)$$

is a $\flat$-equivalence.

Because $X \mapsto X^T$ is already functorial, $X \mapsto X^{1/T}$ becomes functorial for crisp maps.

Thm: If $f :: A \to B$ is between T-null seq. cpt types, then $(\mathrm{Loc}_f X)^T \simeq \mathrm{Loc}_f X^T$, e.g. $\|X\|_n^V \simeq \|X^V\|_n$

for crisp X.

for inf. varieties V.

Lie Grapoids

Def: A type X is **split microlinear** if for any square

$$\begin{tikzcd} V_1 \arrow[r] \arrow[d] & V_3 \arrow[d] \\ V_2 \arrow[r] \arrow[u, bend left, dotted] & V_4 \end{tikzcd}$$

such that

$$\begin{tikzcd} \mathbb{R}^{V_4} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{R}^{V_3} \arrow[d] \\ \mathbb{R}^{V_2} \arrow[r] & \mathbb{R}^{V_1} \end{tikzcd}$$

, then

$$\begin{tikzcd} X^{V_4} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^{V_3} \arrow[d] \\ X^{V_2} \arrow[r] & X^{V_1} \end{tikzcd}$$

.

of infinitesimal varieties

Thm: If G is split microlinear, then BG is too.

Proof:

$$\begin{tikzcd} G^{V_4} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & G^{V_3} \arrow[d] \\ G^{V_2} \arrow[r, two heads] & G^{V_1} \end{tikzcd}$$

and BG^{V_i} is connected, so

$$\begin{tikzcd} BG^{V_4} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & BG^{V_3} \arrow[d] \\ BG^{V_2} \arrow[r] & BG^{V_1} \end{tikzcd}$$

Cor: $B\mathfrak{g} := T_{pt} BG$ has a coherent $\mathbb{R}$-module structure.

Def: A map $f : X \to Y$ is $\mathbb{D}$-étale if it is modally étale for $\mathrm{Loc}_{\mathbb{D}}$.

Thm: A crisp map between ordinary manifolds is $\mathbb{D}$-étale iff it is a local diffeomorphism.

Lem ("good fibrations"): If $f : X \to Y$ satisfies $\forall y : Y. \| F = \mathrm{fib}_f(y) \|$ for a crisp $\mathbb{D}$-null type F, it is $\mathbb{D}$-étale.

Thm: Let $f : X \to Y$ be $\mathbb{D}$-étale.

1) if Y is microlinear, so is X.
2) if f is surjective and X is microlinear, so is Y.

Cor: If Γ is a crisp, $\mathbb{D}$-null higher group and M is microlinear, then $M /\!/ \Gamma$ is microlinear.

i.e. "good orbifolds" are microlinear.

Thm: If $f :: X \to Y$ is surjective and f^*f is $\mathcal{D}$-étale, then f is $\mathcal{D}$-étale. (Works for any "crisply cocontinuous" modality)

Cor: The Rezk completion $\text{よ}\mathcal{G}$ of any étale pregroupoid $\mathcal{G}$ with $\mathcal{G}_0$ microlinear is microlinear.

Proof:

$$\begin{tikzcd}
\mathcal{G}_1 \arrow[r, "t"] \arrow[d, "s"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{G}_0 \arrow[d, two heads, "\text{よ}"] \\
\mathcal{G}_0 \arrow[r, two heads, "\text{よ}"'] & \text{よ}\mathcal{G}
\end{tikzcd}$$

étale

"Étale groupoids are microlinear"

To get to orbifolds, we need to study

Compactness

Def (Dubuc-Penon): A set X is **Dubuc-Penon compact** if

$$\forall A : \mathrm{Prop},\ B : X \to \mathrm{Prop}.\ (\forall x : X.\ A \vee B(x)) \to A \vee (\forall x : X.\ B(x))$$

Def (Penon): A subset $u : X \to \mathrm{Prop}$ is **Penon open** if

$$\forall x : X,\ u(x) \to \forall y : X.\ u(y) \vee (x \neq y)$$ "$u \cup X \setminus \{x\}$ covers X"

Thm (Gabo): Let K be DP-cpt and $u : K \times \mathbb{R} \to \mathrm{Prop}$ be Penon open. If $\forall k : K.\ u(k, x)$, then $\exists \varepsilon > 0$ st $\forall y : \mathbb{R}.\ (x - y)^2 < \varepsilon$, we have $\forall k : K.\ u(k, y)$.

Cor: Any DP-cpt K is **subcountably subcompact**: any subcountable Penon open cover admits a subfinitely enumerable subcover.

Proof: Let $\mathcal{U}_i$ for $i : I \subseteq \mathbb{N}$ be a subcountable cover and consider

$$u(k, x) :\equiv \exists i : I.\ k \in \mathcal{U}_i \wedge (x < 1/i)$$

Cor: Any discrete, DP-cpt subset of a second-countable space is subfinitely enumerable.

Def: A set is properly finite if it is discrete and subfinitely enumerable.

Def: An orbifold is a microlinear type whose types of identifications are properly finite.

fibers are DP-cpt

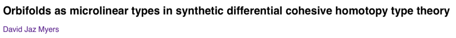
Lem: If $f : M \to N$ is proper and $\mathbb{D}$-étale and M is second-countable, then f's fibers are properly finite.

G_0 and G_1 are ordinary mfds

Thm: The Rezk completion of a crisp, ordinary, proper étale pregroupoid is an orbifold.

Thank You!

More on orbifolds: arXiv: 2205.15887

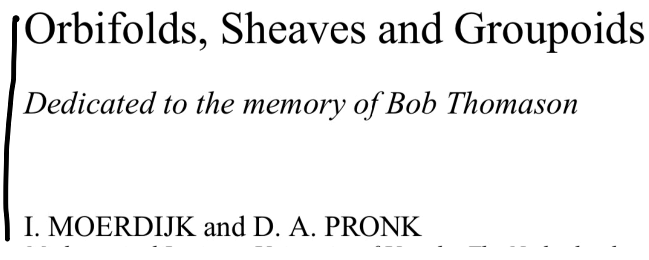

Orbifolds as microlinear types in synthetic differential cohesive homotopy type theory
David Jaz Myers

Some Refs:


Brouwer's fixed-point theorem in real-cohesive homotopy type theory
Michael Shulman


Classifying Types
Egbert Rijke


Modalities in homotopy type theory
Egbert Rijke, Michael Shulman, Bas Spitters


Orbifolds, Sheaves and Groupoids

Dedicated to the memory of Bob Thomason

I. MOERDIJK and D. A. PRONK