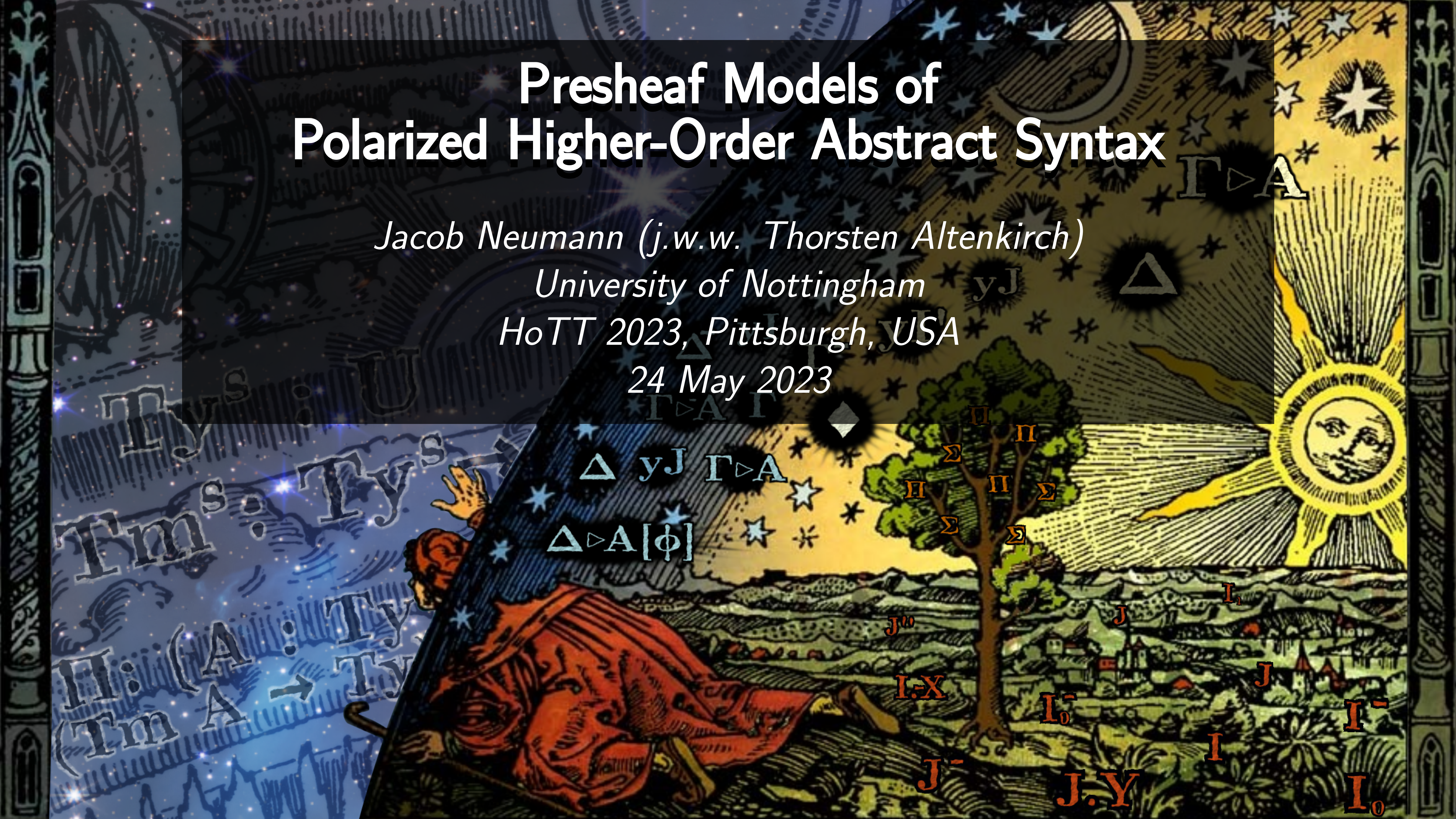

Presheaf Models of Polarized Higher-Order Abstract Syntax

Jacob Neumann (j.w.w. Thorsten Altenkirch)
University of Nottingham

HoTT 2023, Pittsburgh, USA

24 May 2023

Where to find more detail

What I’m interested in:

What I'm interested in:

Directed TT

What I’m interested in:

Directed TT

Higher Observational TT

What I'm interested in:

Directed TT

$\times$

Higher Observational TT?

Key Component

Key Component :

HOAS with polarities

0 Polarized Type Theory

Our approach to type theory: Semantics first!

Defn. A **category with families (CwF)** is a (generalized) algebraic structure, consisting of:

- A category Con of *contexts* and *substitutions*, with a terminal object $\bullet$, the *empty context*
- A presheaf $\mathsf{Ty} : \mathsf{Con}^{\mathsf{op}} \to \mathsf{Set}$ of *types*
- A presheaf $\mathsf{Tm} : (\int \mathsf{Ty})^{\mathsf{op}} \to \mathsf{Set}$ of *terms*
- An operation of *context extension*:

$$\frac{\Gamma : \mathsf{Con} \qquad A : \mathsf{Ty}\ \Gamma}{\Gamma \triangleright A : \mathsf{Con}}$$

so that $\Gamma \triangleright A$ is a 'locally representing object' (in the sense spelled out on the next slide)

For any $I, J :$ Con and any $J :$ Ty Γ,

$$\mathrm{Con}(I, J \triangleright Y) \cong \sum_{j\,:\,\mathrm{Con}(I,J)} \mathrm{Tm}(I, Y[j])$$

natural in I.

Three Important Models of Type Theory

Set	Setoid	Grpd
The Set Model	*The Setoid Model*	*The Groupoid Model*

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[Dyb95, Hof97]	[Hof94, Alt99]	[HS95]

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The Set Model
[Dyb95, Hof97]

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Setoid
The Setoid Model
[Hof94, Alt99]

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Grpd
The Groupoid Model
[HS95]

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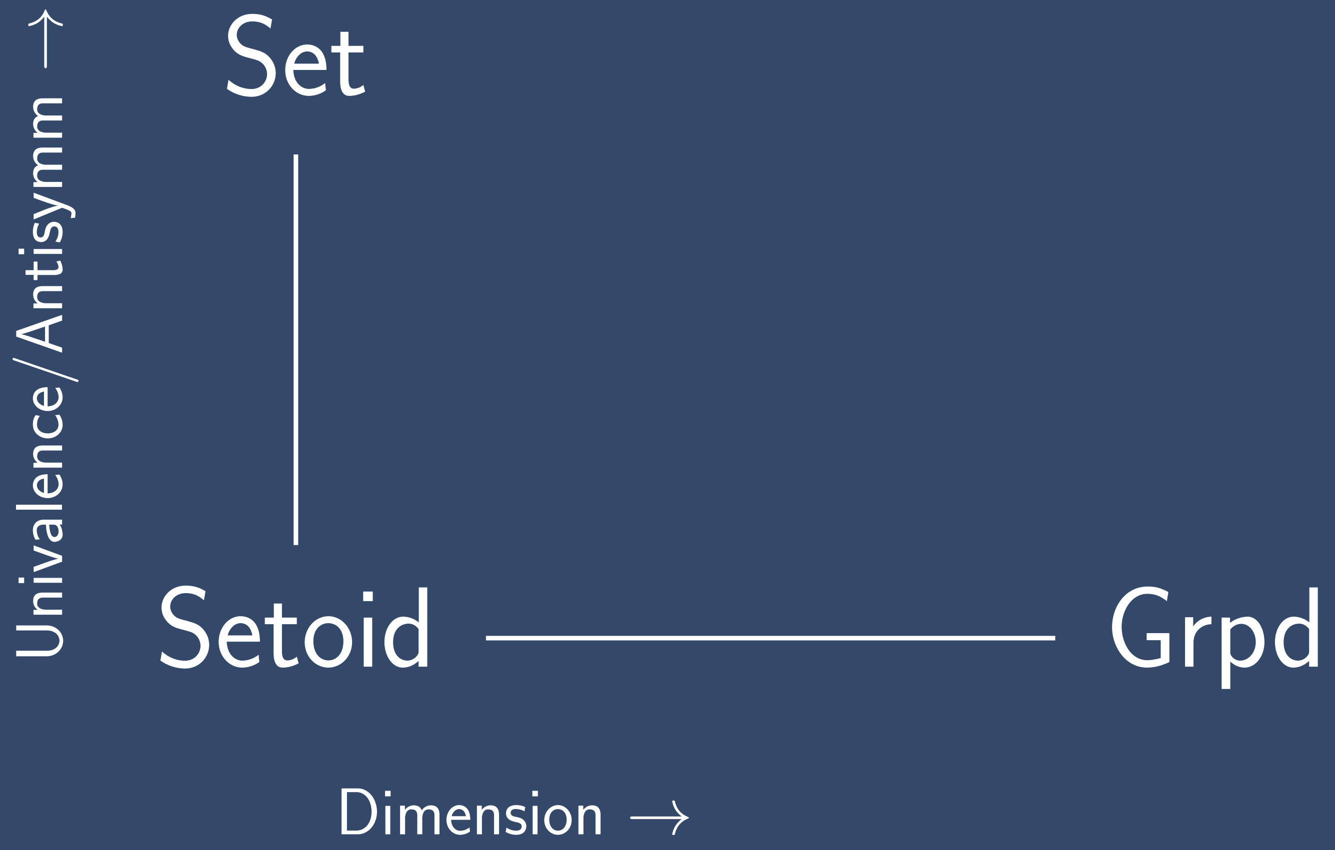
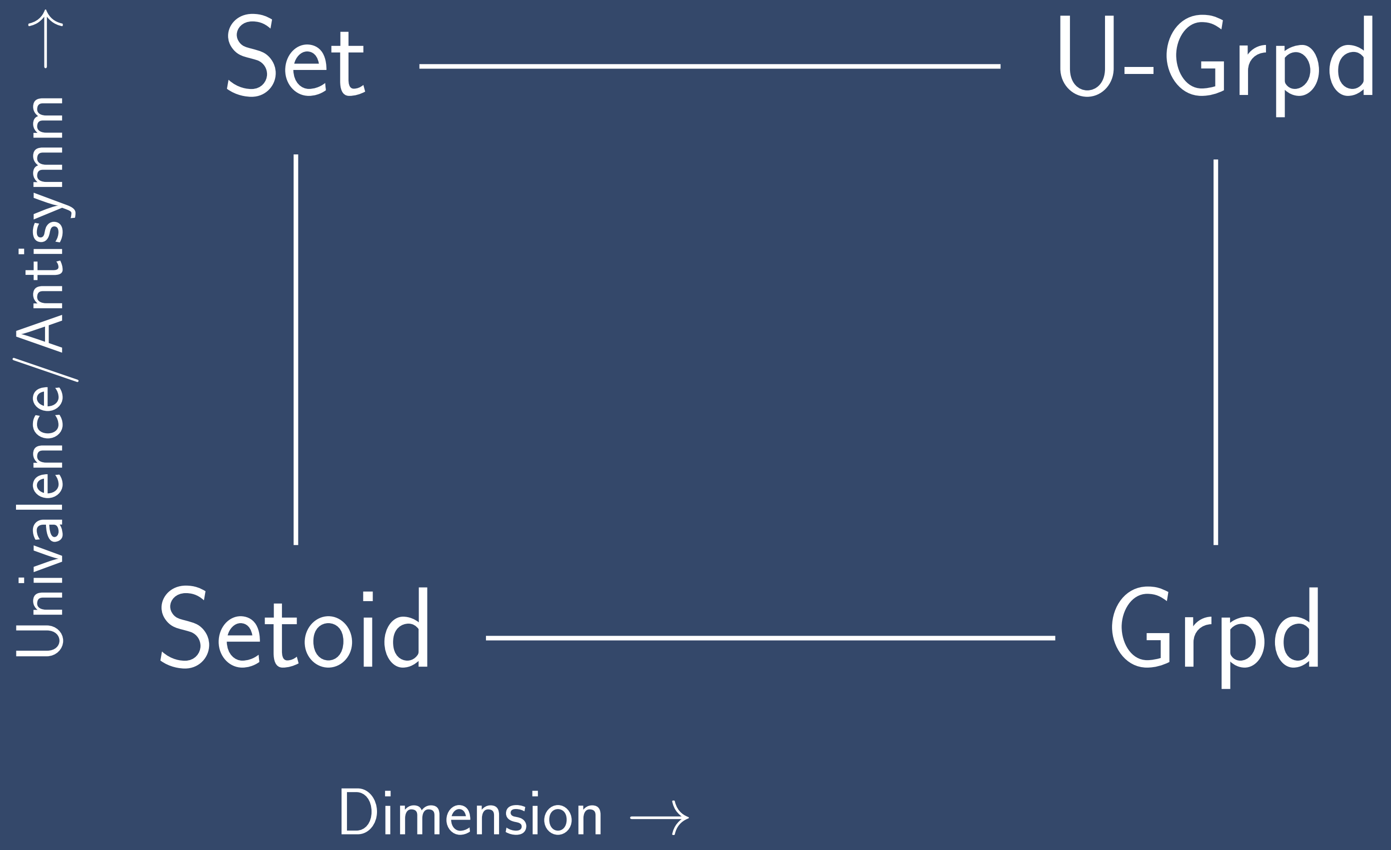
Set
Setoid
Grpd


Set
Setoid
Grpd
Univalence/Antisymm →
Dimension →


Set
U-Grpd
Univalence/Antisymm →
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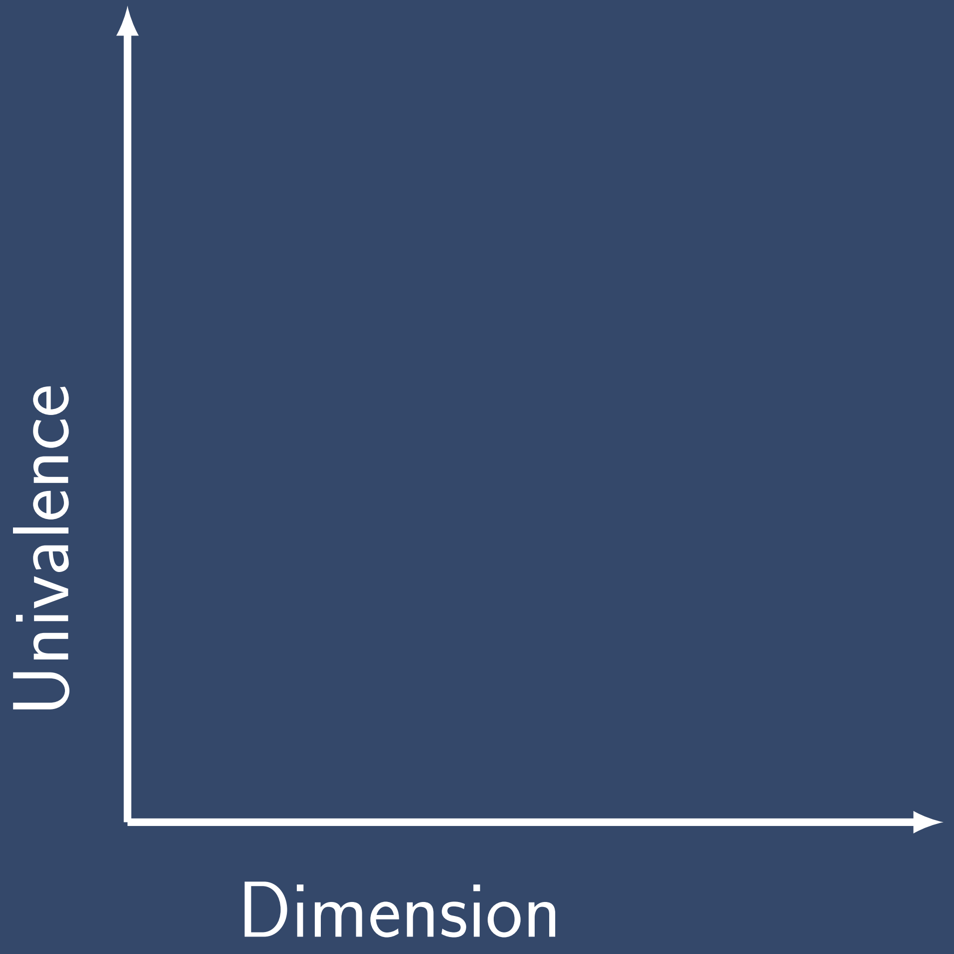
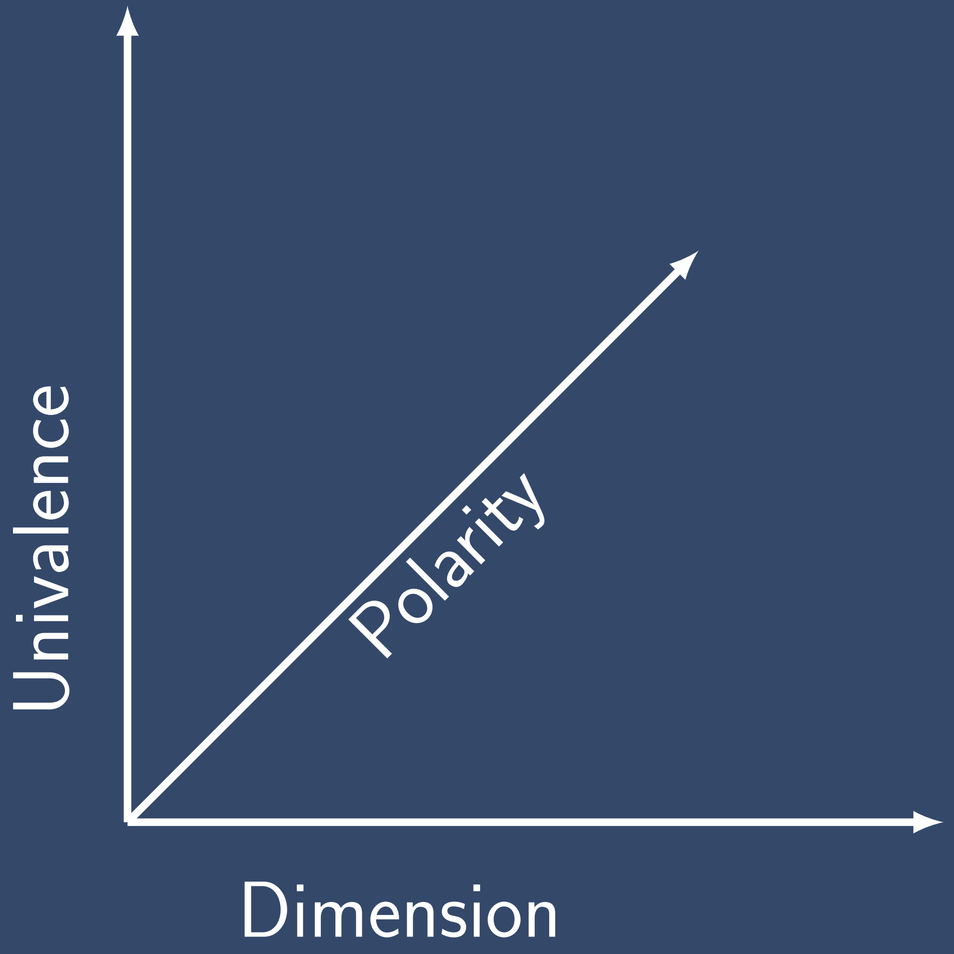
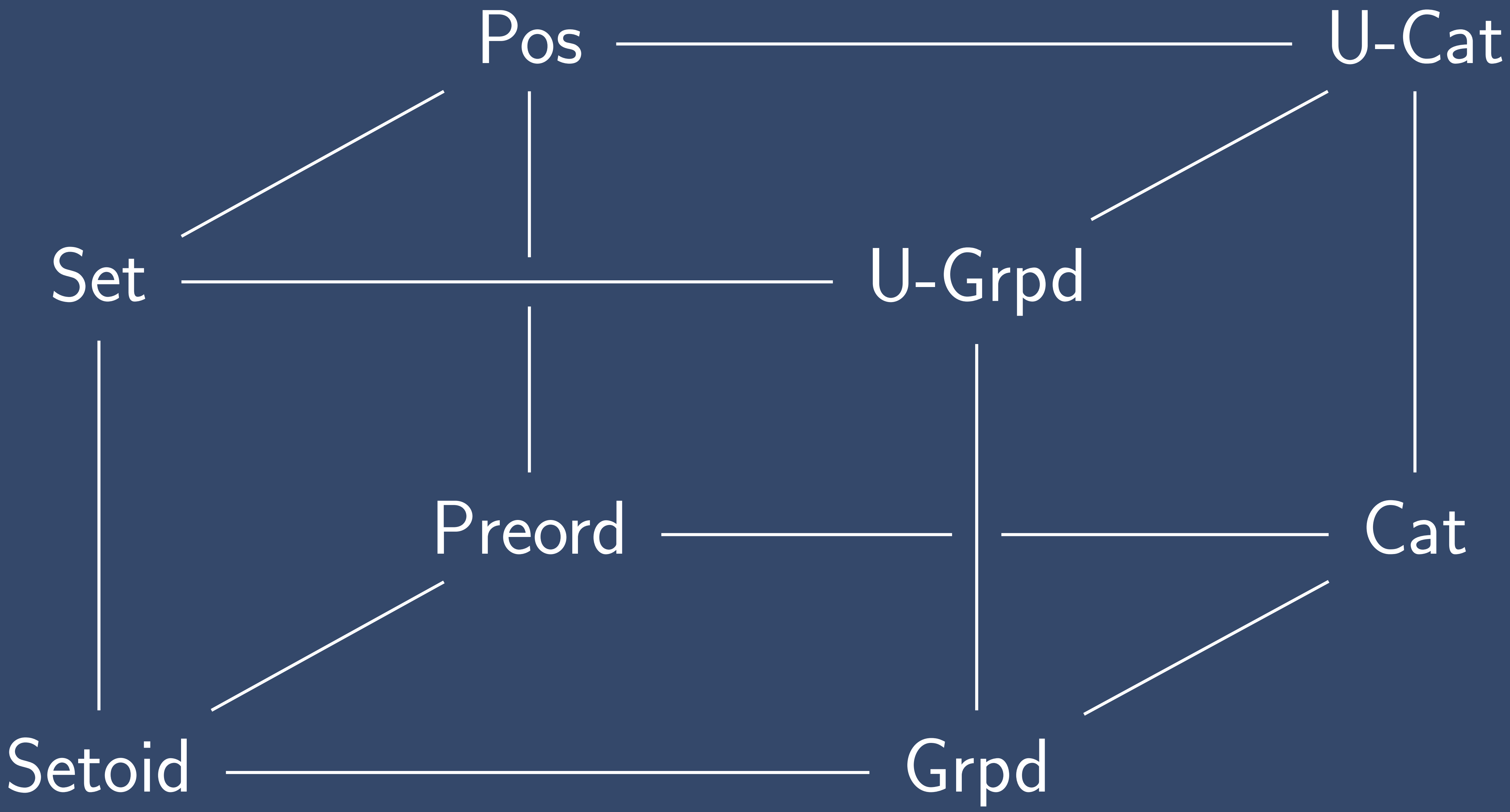
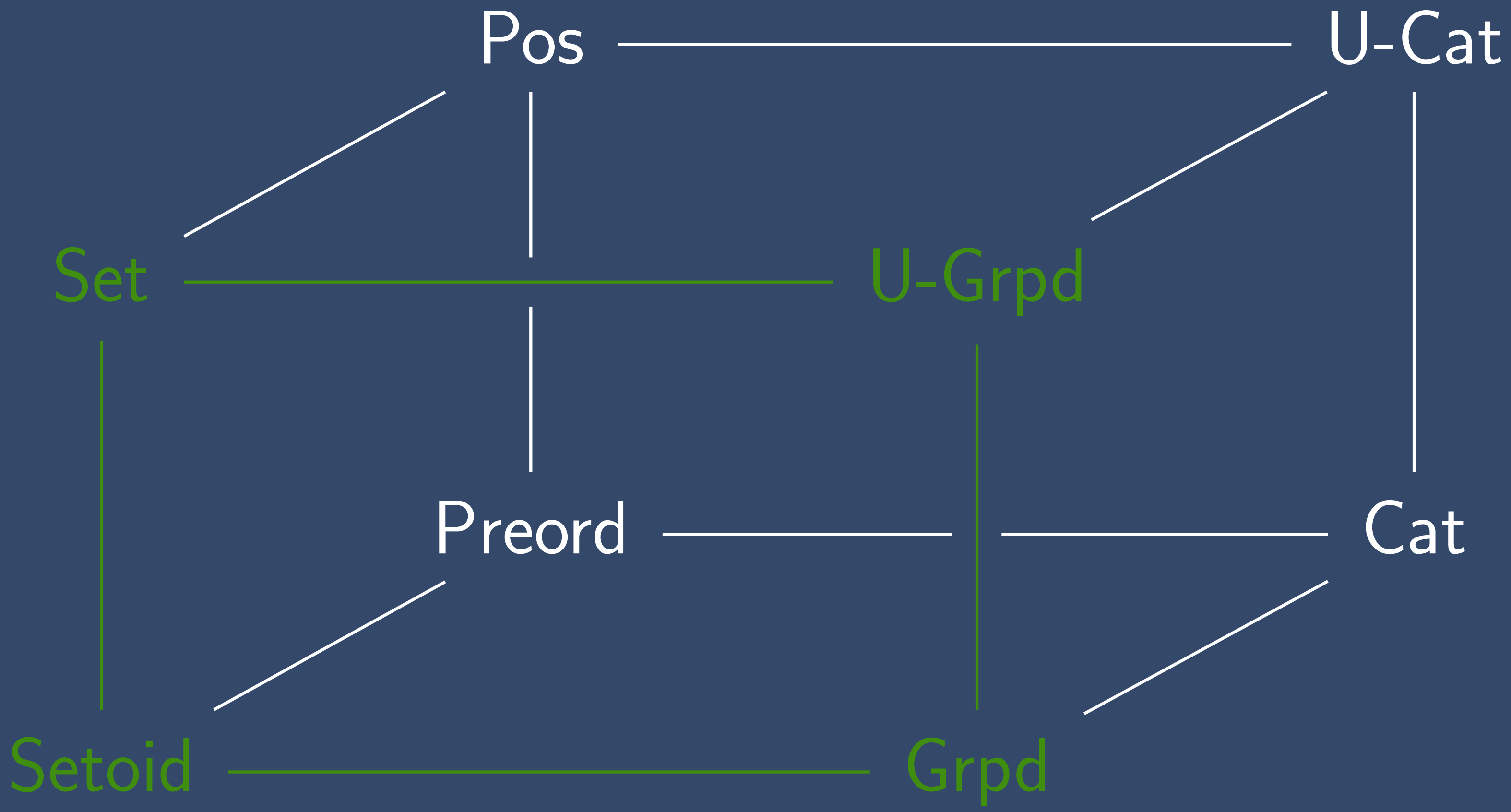

Univalence
Dimension

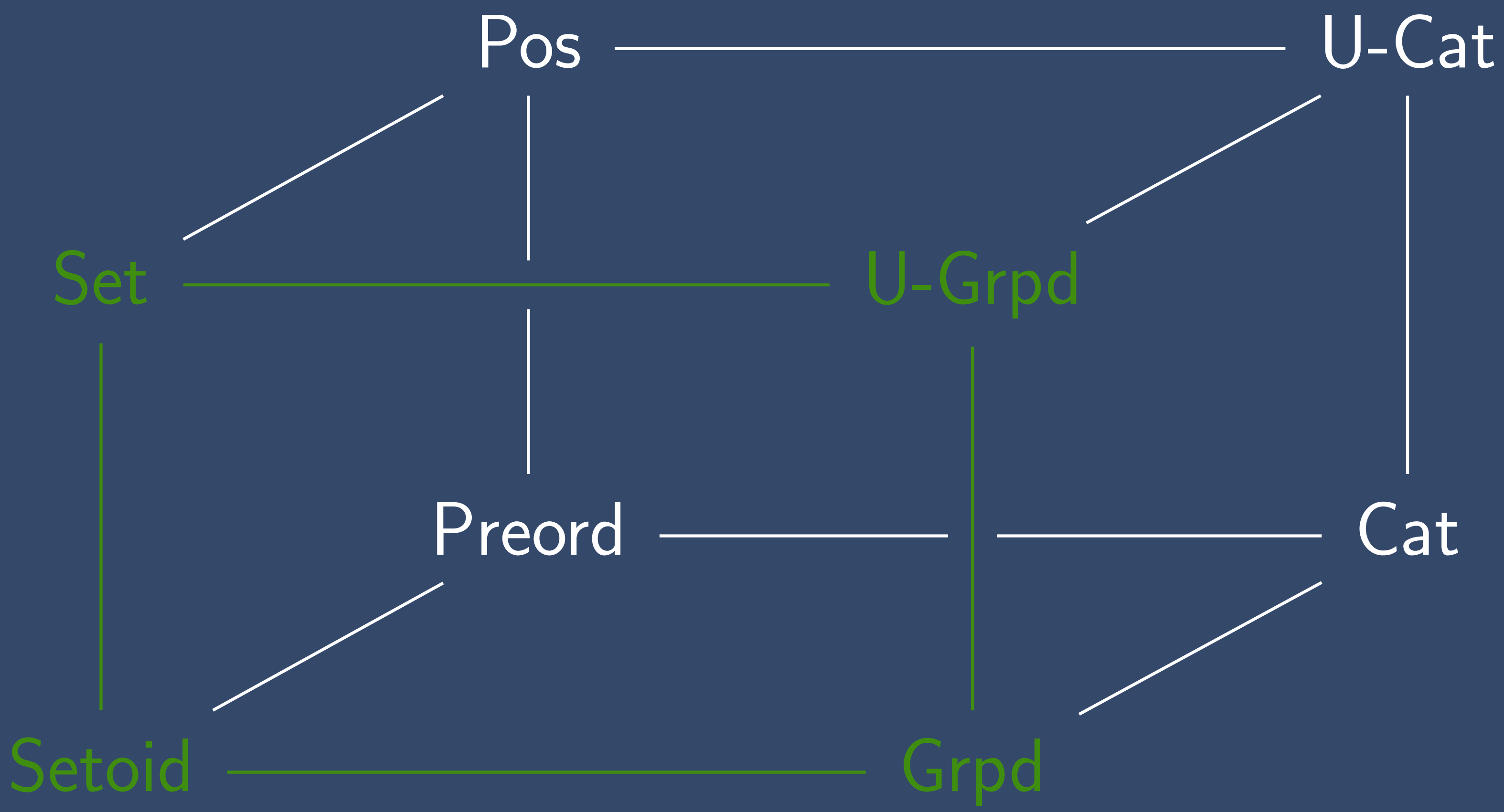

Univalence
Polarity
Dimension


Pos
U-Cat
Set
U-Grpd
Preord
Cat
Setoid
Grpd

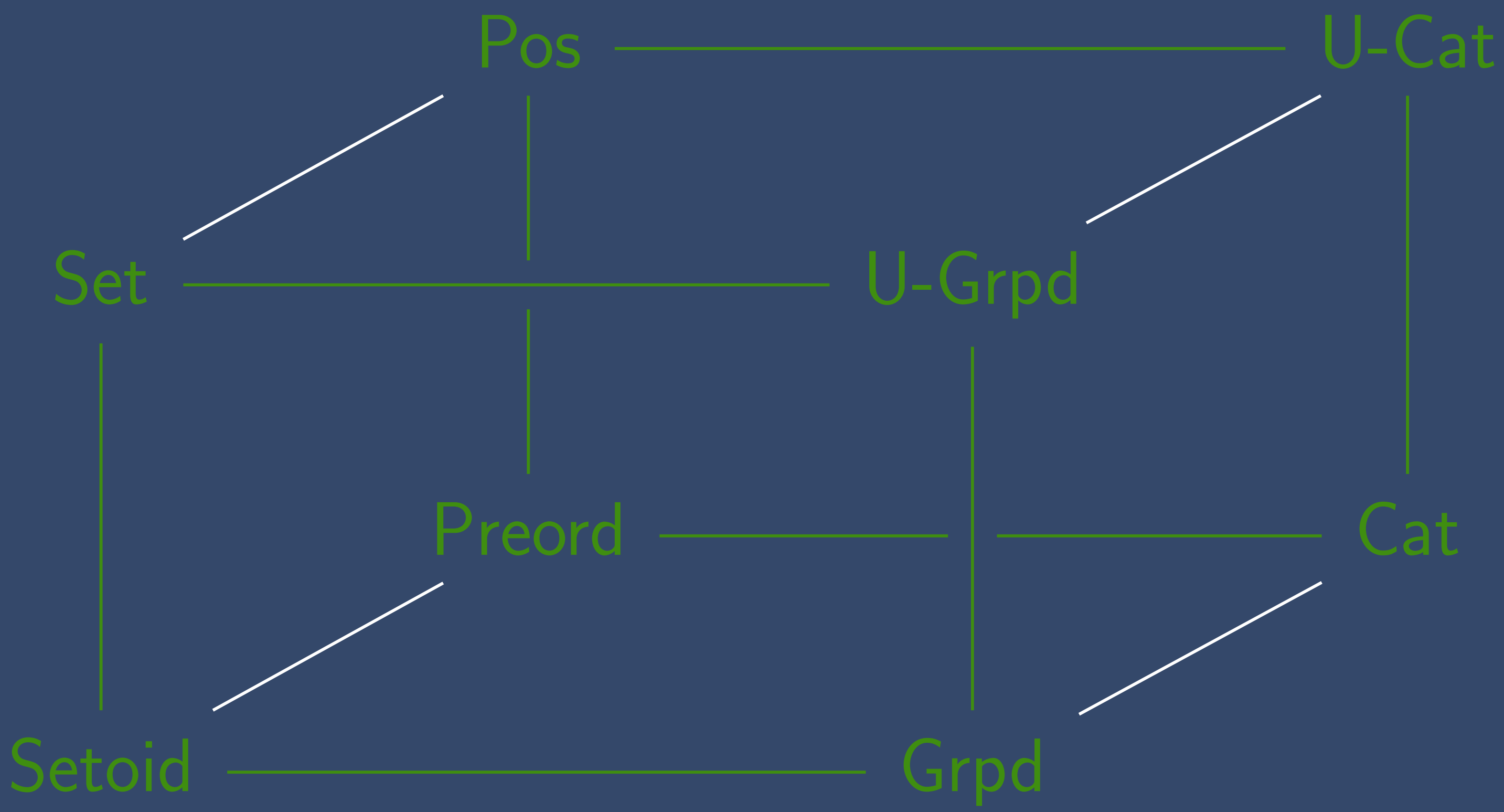


CwFs

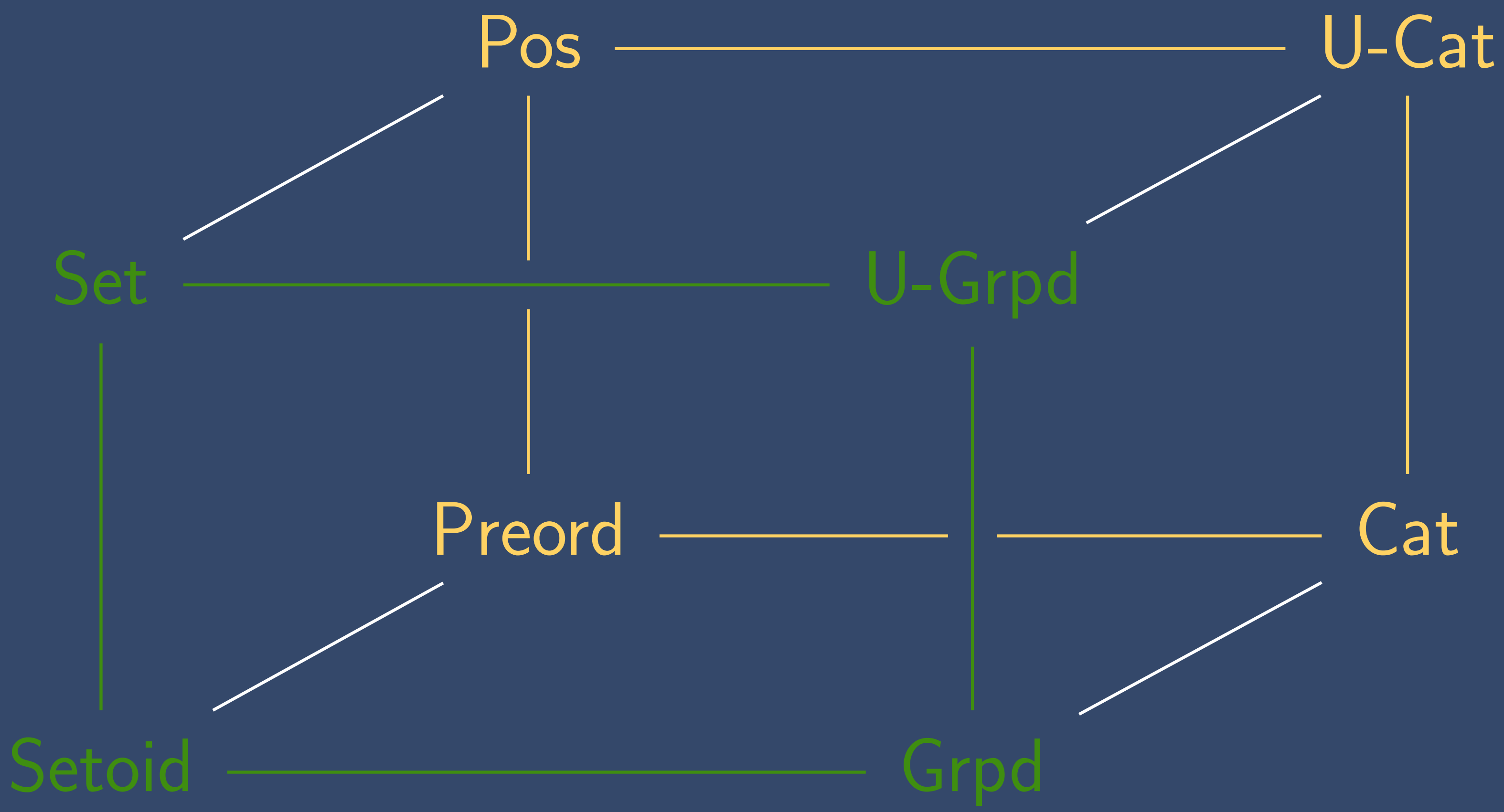
What kinds of models have the back-face structures as contexts?



CwFs



CwFs



CwFs PCwFs

What is a polarized CwF?

Concrete PCwFs

A **(concrete) polarized category with families** is a (generalized) algebraic structure, consisting of:

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- For each J: Con, a function $(\_)^{-}$: Ty $J \to$ Ty J such that $(Y^{-})^{-} = Y$
- Two operations of *context extension*: for s either $+$ or $-$,

$$\frac{J : \mathrm{Con} \quad Y : \mathrm{Ty}(J^{s})}{J \triangleright^{s} Y : \mathrm{Con}}$$

The Local Representability Condition

For any $I, J : \mathsf{Con}$ and any $J : \mathsf{Ty}\, \Gamma^s$,

$$\mathsf{Con}(I, J \triangleright^s Y) \quad \cong \quad \sum_{j\,:\,\mathsf{Con}(I,J)} \mathsf{Tm}(I^s, Y[j^s]^s)$$

natural in I.

The Category Interpretation of Type Theory

The category model of type theory is a PCwF where

- Con is the category of categories and functors
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- Type negation is given by post-composition with the opposite category functor

Context Extension in the Category Model

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Polarized Pi Types

The category model, the preorder model, etc. admit the polarized Π-types of [LH11]:

$$\frac{Y : \mathrm{Ty}\ J^{-} \qquad Z : \mathrm{Ty}(J \triangleright^{-} Y)}{\Pi\ Y\ Z : \mathrm{Ty}\ J}$$

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$$\frac{M : \mathsf{Tm}(J \triangleright^{-} Y, Z)}{(\lambda\ M) : \mathsf{Tm}(J, \Pi\ Y\ Z)}$$

$$\frac{M : \mathsf{Tm}(J, \Pi\ Y\ Z) \qquad N : \mathsf{Tm}(J^{-}, Y^{-})}{(M\ N) : \mathsf{Tm}(J, Z[\overline{N}])}$$

1 Presheaf Semantics of HOAS

Need to explicitly require stability under substitution

Definition 3.15 A CwF supports Π-types if for any two types $\sigma \in Ty(\Gamma)$ and $\tau \in Ty(\Gamma.\sigma)$ there is a type $\Pi(\sigma,\tau) \in Ty(\Gamma)$ and for each $M \in Tm(\Gamma.\sigma,\tau)$ there is a term $\lambda_{\sigma,\tau}(M) \in Tm(\Gamma,\Pi(\sigma,\tau))$ and for each $M \in Tm(\Gamma,\Pi(\sigma,\tau))$ and $N \in Tm(\Gamma,\sigma)$ there is a term $App_{\sigma,\tau}(M,N) \in Tm(\Gamma,\tau\{\overline{M}\})$ such that (the appropriately typed universal closures of) the following equations hold:

$$
\begin{array}{lcll}
App_{\sigma,\tau}(\lambda_{\sigma,\tau}(M),N) & = & M\{\overline{N}\} & \Pi\text{-C} \\
\Pi(\sigma,\tau)\{f\} & = & \Pi(\sigma\{f\},\tau\{\mathsf{q}(f,\sigma)\}) \in Ty(\mathrm{B}) & \Pi\text{-S} \\
\lambda_{\sigma,\tau}(M)\{f\} & = & \lambda_{\sigma\{f\},\tau\{\mathsf{q}(f,\sigma)\}}(M\{\mathsf{q}(f,\sigma)\}) & \lambda\text{-S} \\
App_{\sigma,\tau}(M,N)\{f\} & = & App_{\sigma\{f\},\tau\{\mathsf{q}(f,\sigma)\}}(M\{f\},N\{f\}) & App\text{-S}
\end{array}
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From [Hof97, 3.3]

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\end{array}
$$

} annoying! (Π-S, λ-S, App-S)

From [Hof97, 3.3]

Solution: Use higher-order abstract syntax!

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(and interpret it in a presheaf category!)

Steps

1. Presheaf Model

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2. Lift Grothendieck Universe(s) [HS99]

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3. Higher-Order Abstract Syntax [Hof99]

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Claim This model of type theory supports Π-types

Lifting Grothendieck Universes

We want a universe, i.e. a closed type $\mathbf{U}$ such that

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\end{aligned}$$

So just define $\mathbf{U}\ I$ to be the set of presheaves on $\mathbb{C}/I$.

What if $\mathbb{C}$ is *itself* a CwF?

Key Idea: Talk about the "ground" CwF structure using the presheaf CwF structure

Semantics	HOAS

Higher-Order Abstract Syntax

Semantics	**HOAS**
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Semantics	**HOAS**
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$\mathrm{Tm} \colon \widehat{\mathrm{Tm}}(\blacklozenge, \mathrm{Ty} \Rightarrow \mathbf{U})$	$\mathrm{Tm} \colon \mathrm{Ty} \to \mathbf{U}$
$\ldots$	$\Pi \colon (A \colon \mathrm{Ty}) \to (\mathrm{Tm}\ A \to \mathrm{Ty}) \to \mathrm{Ty}$

2 Polarized HOAS

Problem: How do we talk about operations on contexts, after we've abstracted them away?

Hint: we don't need context- and type-negation to be independent

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$$\mathsf{Con}(I, J \triangleright^s Y) \quad \cong \quad \sum_{j : \mathsf{Con}(I,J)} \mathsf{Tm}(I^s, Y[j^s]^s)$$

Hint: we don't need context- and type-negation to be independent

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Defining Ty^{-}

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$$\begin{aligned} \mathsf{Ty}^- &: \mathsf{Con}^{\mathsf{op}} \to \mathsf{Set} \\ \mathsf{Ty}^- J &:= \mathsf{Ty}(J^-) \\ Y[j] &:= Y[j^-] \qquad\qquad (j : \mathsf{Con}(I, J),\ Y : \mathsf{Ty}^-\ J) \end{aligned}$$

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\end{aligned}$$

$$\begin{aligned}
\mathsf{Tm}^- &: \textstyle\int \mathsf{Ty}^- \to \mathsf{Set} \\
\mathsf{Tm}^-(J, Y) &:= \mathsf{Tm}(J^-, Y^-) \\
M[j] &:= M[j^-] && (j : \mathsf{Con}(I, J),\ M : \mathsf{Tm}^-(J, Y))
\end{aligned}$$

Revisited: we don't need context- and type-negation to be independent

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$$\mathrm{Con}(I, J \rhd^{s} Y) \quad \cong \quad \sum_{j\,:\,\mathrm{Con}(I,J)} \mathrm{Tm}^{s}(I, Y[j])$$

Revisited: we don't need context- and type-negation to be independent

$$\mathrm{Con}(I, J \triangleright^s Y) \quad \cong \quad \sum_{j\,:\,\mathrm{Con}(I,J)} \mathrm{Tm}^s(I, Y[j])$$

$$\frac{M : \mathrm{Tm}(J, \Pi\ Y\ Z) \qquad N : \mathrm{Tm}^{-}(J, Y)}{(M\ N) : \mathrm{Tm}(J, Z[\overline{N}])}$$

Defn. An **abstractly polarized CwF** is a category Con with a terminal object $\bullet$ and *two* CwF structures

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Question What more should be added to this definition?

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Question What more should be added to this definition?

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- ??

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- When adapting [ABK+21]'s Agda formalization of the setoid model, it is very straightforward to define it as an abstract PCwF but proving much more difficult to do as a concrete PCwF

Idea The presheaf model over an abstract PCwF gives us a 2-level type theory: the inner layer polarized, the outer unpolarized

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$\ldots$	$\Pi : (A : \mathrm{Ty}^-) \to (\mathrm{Tm}^-\ A \to \mathrm{Ty}) \to \mathrm{Ty}$

Further Topics of Study

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- Core types, neutral-zoned contexts

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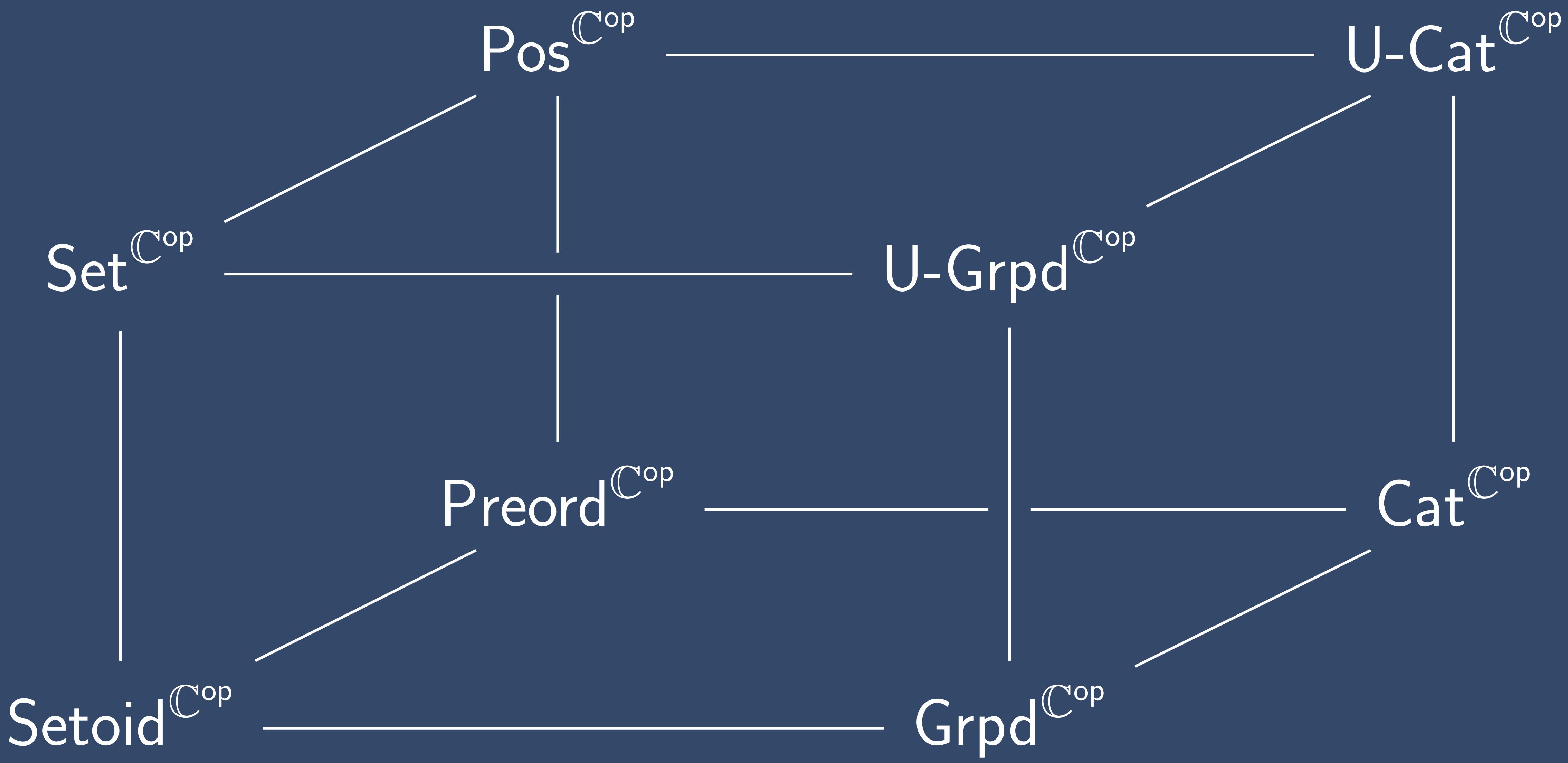
- Connections to other varieties of polarized/directed TT
- Polarizing both layers


$\mathbf{Pos}^{\mathbb{C}^{op}}$
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$\mathbf{Set}^{\mathbb{C}^{op}}$
$\text{U-}\mathbf{Grpd}^{\mathbb{C}^{op}}$
$\mathbf{Preord}^{\mathbb{C}^{op}}$
$\mathbf{Cat}^{\mathbb{C}^{op}}$
$\mathbf{Setoid}^{\mathbb{C}^{op}}$
$\mathbf{Grpd}^{\mathbb{C}^{op}}$

[ABK+21] Thorsten Altenkirch, Simon Boulier, Ambrus Kaposi, Christian Sattler, and Filippo Sestini.
Constructing a universe for the setoid model.
In *FoSSaCS*, pages 1–21, 2021.

[Alt99] Thorsten Altenkirch.
Extensional equality in intensional type theory.
In *Proceedings. 14th Symposium on Logic in Computer Science (Cat. No. PR00158)*, pages 412–420. IEEE, 1999.

[Awo18] Steve Awodey.
Natural models of homotopy type theory.
Mathematical Structures in Computer Science, 28(2):241–286, 2018.

[Dyb95] Peter Dybjer.
Internal type theory.
In *International Workshop on Types for Proofs and Programs*, pages 120–134. Springer, 1995.

[Hof94] Martin Hofmann.
Elimination of extensionality in Martin-Löf type theory.
In *Types for Proofs and Programs: International Workshop TYPES'93 Nijmegen, The Netherlands, May 24–28, 1993 Selected Papers*, pages 166–190. Springer, 1994.

[Hof97] Martin Hofmann.
Syntax and semantics of dependent types.
In *Extensional Constructs in Intensional Type Theory*, pages 13–54. Springer, 1997.

[Hof99] Martin Hofmann.
Semantical analysis of higher-order abstract syntax.
In *Proceedings. 14th Symposium on Logic in Computer Science (Cat. No. PR00158)*, pages 204–213. IEEE, 1999.

[HS95] Martin Hofmann and Thomas Streicher.
The groupoid interpretation of type theory.
Twenty-five years of constructive type theory (Venice, 1995), 36:83–111, 1995.

[HS99] Martin Hofmann and Thomas Streicher.
Lifting grothendieck universes.
Unpublished note, 199:3, 1999.

[LH11] Daniel R Licata and Robert Harper.
2-dimensional directed dependent type theory.
2011.

Thank you!!