The unifying power of modal type theory

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The proliferation of type theories

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Is there a unified type theory that includes all these examples?

Outline

What is modal type theory?

A modal type theory consists of

1. One or more ordinary type theories.
2. New unary type formers acting on or between them.
   (Higher-ary type formers make a “substructural” type theory.)
3. Functions relating these type formers and their composites.

Licata, Shulman, Riley, “A Fibrational Framework for Substructural and Modal Logics”, FSCD'17

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Accordingly, it is specified by a 2-category $\mathcal{M}$, with

1. Objects $p, q, r, \dots$ called modes.
2. Morphisms $\mu : p \to q, \dots$ called modalities.
3. 2-cells $\alpha : \mu \Rightarrow \nu, \dots$ which today I will call laws.

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And it should have semantics in a (pseudo) 2-functor $\mathcal{M} \to \mathcal{C}at$:

1. Each mode represents a category.
2. Each modality represents a functor.
3. Each law represents a natural transformation.

Licata, Shulman, Riley, "A Fibrational Framework for Substructural and Modal Logics", FSCD'17

Modal dependence

- Each mode has its own ordinary type theory.
- For a p-type A and a q-type B, with $\mu : p \to q$,

  $$f : (x :^{\mu} A) \to B$$

  is a function associating, to any x in A, an element of B that depends on x through μ.
- Ordinary $(x : A) \to B$ coincides with $(x :^{1_p} A) \to B$.

Positive modalities

A modality $\mu : p \to q$ maps a p-type A to a q-type $\mu \boxdot A$, internalizing μ-dependence with a universal property:

$$(x :^{\mu} A) \to B \quad \simeq \quad (y : \mu \boxdot A) \to B$$

- Semantically, $x :^{\mu} A$ and $y : \mu \boxdot A$ are equivalent.
- Syntactically, we have a constructor $\mathsf{mod} : (x :^{\mu} A) \to \mu \boxdot A$ with an induction principle that any $y : \mu \boxdot A$ can be assumed to be $\mathsf{mod}(x)$ for some $x :^{\mu} A$.

Outline

Cohesive type theory

Example (Spatial type theory)

- One mode t.
- Modalities $\flat : t \to t$ and $\sharp : t \to t$.
- $\flat$ is an idempotent comonad, $\sharp$ is an idempotent monad.
- An adjunction $\flat \dashv \sharp$.

Semantics in topological* spaces.

- $\flat A = A$ retopologized discretely
- $\sharp A = A$ retopologized indiscretely

Shulman, "Brouwer's fixed-point theorem in real-cohesive homotopy type theory", 1509.07584

Cohesive type theory

Example (Cohesive type theory)

- One mode t.
- Modalities $\flat : t \to t$ and $\sharp : t \to t$ and $\pi_0 : t \to t$.
- $\flat$ is an idempotent comonad, $\sharp$ and π_0 are idempotent monads.
- Adjunctions $\pi_0 \dashv \flat \dashv \sharp$.

Semantics in locally connected topological* spaces.

- $\flat A = A$ retopologized discretely
- $\sharp A = A$ retopologized indiscretely
- $\pi_0 A =$ the set of connected components of A, discretely

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Cohesive type theory

Example (Cohesive homotopy type theory)

- One mode t.
- Modalities $\flat : t \to t$ and $\sharp : t \to t$ and $\int : t \to t$.
- $\flat$ is an idempotent comonad, $\sharp$ and $\int$ are idempotent monads.
- Adjunctions $\int \dashv \flat \dashv \sharp$.

Semantics in locally contractible topological* ∞-groupoids.

- $\flat A = A$ retopologized discretely
- $\sharp A = A$ retopologized indiscretely
- $\int A =$ the shape (fundamental ∞-groupoid) of A, discretely

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Semantics in locally contractible topological* ∞-groupoids.

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Other semantics include

- Smooth ∞-groupoids (SDG — cf. Myers' talk Monday)
- Simplicial ∞-groupoids (shape is geometric realization)

Shulman, "Brouwer's fixed-point theorem in real-cohesive homotopy type theory", 1509.07584

Commuting cohesions

As in Riley's talk Monday:

Example

- One mode.
- Endo-modalities $\int_{\heartsuit}, \flat_{\heartsuit}, \sharp_{\heartsuit}, \int_{\clubsuit}, \flat_{\clubsuit}, \sharp_{\clubsuit}$.
- $\flat_{\heartsuit}, \flat_{\clubsuit}$ are idemp. comonads, $\int_{\heartsuit}, \sharp_{\heartsuit}, \int_{\clubsuit}, \sharp_{\clubsuit}$ are idemp. monads.
- Adjunctions $\int_{\heartsuit} \dashv \flat_{\heartsuit} \dashv \sharp_{\heartsuit}$ and $\int_{\clubsuit} \dashv \flat_{\clubsuit} \dashv \sharp_{\clubsuit}$.
- $\flat_{\heartsuit} \circ \flat_{\clubsuit} = \flat_{\clubsuit} \circ \flat_{\heartsuit}$, etc.

Should have semantics in simplicial topological ∞-groupoids.

Myers, Riley, "Commuting Cohesions", 2301.13780

More single-mode examples

- Crisp type theory: One idempotent comonad $\flat$. Semantics in "global sections" of any connected topos.
- Synthetic stable homotopy theory: a self-adjoint idempotent monad/comonad $\natural$. Semantics in parametrized spectra.
- Synthetic guarded domain theory: an idempotent comonad $\Box$ and a "later" endofunctor $\triangleright$. Semantics in the "topos of trees" $\mathrm{Set}^{\omega^{\mathrm{op}}}$.
- Directed type theory: an idempotent comonad "core" and an involution "op". Semantics in $\mathcal{C}at$ or $\infty\mathcal{C}at$.

Licata, Orton, Pitts, Spitters, "Internal Universes in Models of Homotopy Type Theory", 1801.07664
Riley, Finster, Licata, "Synthetic Spectra via a Monadic and Comonadic Modality", 2102.04099
Cisinski, Nguyen, Walde, "Univalent Directed Type Theory", CMU Seminar 2023

Two-level type theory

As in Uskuplu's talk. Not originally written modally, but it can be:

- Two modes: e for exo-types, f for fibrant types.
- Modalities $\alpha : f \to e$ and $\beta : e \to f$ forming an isomorphism.
- $\alpha \boxdot X$ is the "coercion" $c(X)$ from fibrant types to exo-types. We omit $\beta \boxdot X$, since fibrant replacement is inconsistent.

Voevodsky, "A simple type system with two identity types", IAS 2013
Annenkov, Capriotti, Kraus, Sattler, "Two-Level Type Theory and Applications", 1705.03307

Indexed type theory

If $p : \mathcal{E} \to \mathcal{B}$ is a fibration whose fibers have terminal objects, we have an adjunction

$$\mathcal{B} \underset{\text{terminals}}{\overset{p}{\rightleftarrows}} \mathcal{E}$$

This modal type theory is similar to Isaev's indexed type theories.

Isaev, "Indexed type theories", 1806.08038

Identity types

Identity types can also be considered a "unary type former":

$$A \quad \mapsto \quad \mathsf{Id}_A$$

The only difference is that Id_A is indexed by $A \times A$.

So we should have one mode p, with one modality $\iota : p \to p$.

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...and of course we also have $\iota \circ \iota$, etc.

So the (monoidal) hom-category $\mathcal{M}(p, p)$ is some cube category.

Cavallo, "Higher Inductive Types and Internal Parametricity for Cubical Type Theory", Ph.D. Thesis

Combining type theories

If $\mathcal{M}$ and $\mathcal{N}$ are 2-categories, so is $\mathcal{M} \times \mathcal{N}$.

- cohesion $\times$ cohesion $=$ two commuting cohesions
- cohesion $\times$ cubes
- 2LTT $\times$ cubes
- directed $\times$ 2LTT (as in Neumann's talk)

Also gives a framework for more refined combinations,
e.g. 2LTT with only the inner layer being cubical or directed.

Outline

Dealing with modal contexts

Question #1

What kind of thing can a modal function be applied to?

E.g. the constructor $\mathsf{mod} : (x :^{\mu} A) \to \mu \boxdot A$ requires a rule

$$\frac{\mathbf{?} \vdash M : A}{\Gamma \vdash \mathsf{mod}(M) : \mu \boxdot A}$$

If $\mu : p \to q$, then Γ is a q-context, but **?** must be a p-context!

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Question #2

When can we use a modal variable $x :^{\mu} A$?

$(\Gamma, x :^{\mu} A)$ is a q-context, but A is a p-type, so we have no type in context $(\Gamma, x :^{\mu} A)$ for x to belong to.

Introducing context division

We need to "cancel out" the μ annotation on x, to use it.

First idea

Define the *p*-context $\Gamma/_{\mu}$ (also written $\Gamma.\text{🔒}_{\mu}$ or $\Gamma.\{\mu\}$ or $\mu\backslash\Gamma$) by:

- For every $x :^{\mu} A$ in Γ, we have $x : A$ in $\Gamma/_{\mu}$.
- Omit all the other variables.

Then the rule for mod is

$$\frac{\Gamma/_{\mu} \vdash M : A}{\Gamma \vdash \mathsf{mod}(M) : \mu \boxdot A}$$

This is the correct rule, but our definition of $\Gamma/_{\mu}$ needs work.

Laws in context division

$\alpha : \mu \Rightarrow \nu$ should induce $\mu \boxdot A \to \nu \boxdot A$, that is $x :^{\mu} A \vdash \mathbf{?} : \nu \boxdot A$.

$$\frac{(x :^{\mu} A)/\nu \vdash \mathbf{?} : A}{x :^{\mu} A \vdash \mathbf{?} : \nu \boxdot A}$$

If we omit x from $(x :^{\mu} A)/\nu$ since $\mu \neq \nu$, we have nothing left.

Second idea

Define Γ/ν by

- For $x :^{\mu} A$ in Γ, if there is $\alpha : \mu \Rightarrow \nu$, we have $x : A$ in Γ/ν.
- Omit all the other variables.

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Partial cancellation in context division

Modalities are functorial; we should have $(\nu \circ \mu)\boxdot A \to \nu\boxdot(\mu\boxdot A)$, that is $x :^{\nu\circ\mu} A \vdash \mathbf{?} : \nu\boxdot(\mu\boxdot A)$.

$$\cfrac{\cfrac{(x :^{\nu\circ\mu} A)/\nu/\mu \vdash \mathbf{?} : A}{(x :^{\nu\circ\mu} A)/\nu \vdash \mathbf{?} : \mu\boxdot A}}{x :^{\nu\circ\mu} A \vdash \mathbf{?} : \nu\boxdot(\mu\boxdot A)}$$

If x disappears in $(x :^{\nu\circ\mu} A)/\nu$ since $\nu \circ \mu \not\Rightarrow \nu$, there's nothing left.

Third idea

Define Γ/ν by

- For $x :^{\mu} A$ in Γ, if $\alpha : \mu \Rightarrow \nu \circ \varrho$, we have $x :^{\varrho} A$ in Γ/ν.
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Define $\Gamma/_{\nu}$ by

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- Omit all the other variables.

Partial cancellation in context division

Modalities are functorial; we should have $(\nu \circ \mu)\boxdot A \to \nu\boxdot(\mu\boxdot A)$, that is $x :^{\nu\circ\mu} A \vdash \textbf{?} : \nu\boxdot(\mu\boxdot A)$.

$$\dfrac{\dfrac{x : A \vdash x : A}{x :^{\mu} A \vdash \mathsf{mod}_{\mu}(x) : \mu\boxdot A}}{x :^{\nu\circ\mu} A \vdash \textbf{?} : \nu\boxdot(\mu\boxdot A)}$$

If x disappears in $(x :^{\nu\circ\mu} A)/\nu$ since $\nu \circ \mu \not\Rightarrow \nu$, there's nothing left.

Third idea

Define Γ/ν by

- For $x :^{\mu} A$ in Γ, if $\alpha : \mu \Rightarrow \nu \circ \varrho$, we have $x :^{\varrho} A$ in Γ/ν.
- Omit all the other variables.

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$$\dfrac{\dfrac{x : A \vdash x : A}{x :^{\mu} A \vdash \mathsf{mod}_{\mu}(x) : \mu\boxdot A}}{x :^{\nu\circ\mu} A \vdash \mathsf{mod}_{\nu}(\mathsf{mod}_{\mu}(x)) : \nu\boxdot(\mu\boxdot A)}$$

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- Omit all the other variables.

Parallel laws in context division

Two laws $\alpha, \beta : \mu \Rightarrow \nu$ should give two terms $x :^{\mu} A \vdash \mathbf{?} : \nu \boxdot A$.

Fourth idea (∼LSR)

Define Γ/ν by

- Replace $x :^{\mu} A$ in Γ by a variable $x^{\alpha} :^{\varrho} A$ for each pair (ϱ, α) with $\alpha : \mu \Rightarrow \nu \circ \varrho$.

Given $\alpha, \beta : \mu \Rightarrow \nu$, we have $(x :^{\mu} A)/\nu = (x^{\alpha} : A, x^{\beta} : A)$, and

$$\frac{x^{\alpha} : A, x^{\beta} : A \vdash x^{\alpha} : A}{x :^{\mu} A \vdash \mathsf{mod}(x^{\alpha}) : \nu \boxdot A} \qquad \frac{x^{\alpha} : A, x^{\beta} : A \vdash x^{\beta} : A}{x :^{\mu} A \vdash \mathsf{mod}(x^{\beta}) : \nu \boxdot A}$$

Licata, Shulman, Riley, "A Fibrational Framework for Substructural and Modal Logics", FSCD'17

Too much choice in context division

At last we have enough variables... but actually we have too many.

$$\mathcal{M} = \quad p \underset{\nu}{\overset{\mu}{\rightrightarrows}} q \xrightarrow{\varrho} r \qquad (\Downarrow\alpha : \mu \Rightarrow \nu)$$

Then $1_{\varrho\circ\mu} : (\varrho \circ \mu) \Rightarrow \varrho \circ \mu$ and $\varrho \triangleleft \alpha : (\varrho \circ \mu) \Rightarrow \varrho \circ \nu$, so

$$(x :^{\sigma} A)/_{\varrho} = (x^{1_{\varrho\circ\mu}} :^{\mu} A,\ x^{\varrho\triangleleft\alpha} :^{\nu} A)$$
$$(x :^{\sigma} A)/_{\varrho}/_{\nu} = (x^{1_{\varrho\circ\mu},\alpha} : A,\ x^{\varrho\triangleleft\alpha,1_{\nu}} : A)$$

We get two maps $(x :^{\varrho\circ\mu} A) \to \varrho\boxdot(\nu\boxdot A)$ instead of just one.

Thus, LSR imposes equations between canonical forms such as

$$\mathrm{mod}(x^{1_{\varrho\circ\mu},\alpha}) \equiv \mathrm{mod}(x^{\varrho\triangleleft\alpha,1_{\nu}}).$$

Multimodal Type Theory

Final idea

Delay the choice of α until the time of use of the variable.
Division is a constructor of contexts, not an operation on them.

Contexts defined inductively from empty, variables, and divisions:

$$\frac{}{\diamond_p \ \mathsf{ctx}_p} \qquad \frac{\Gamma \ \mathsf{ctx}_q \qquad \mu : p \to q \qquad \Gamma/_\mu \vdash A \ \mathsf{type}_p}{\Gamma, (x :^\mu A) \ \mathsf{ctx}_q}$$

$$\frac{\Gamma \ \mathsf{ctx}_q \qquad \mu : p \to q}{\Gamma/_\mu \ \mathsf{ctx}_p}$$

Now we choose a law when we use a variable, e.g.

$$\frac{\alpha : \mu \Rightarrow \nu \circ \varrho}{\Gamma, (x :^\mu A) \ /_\nu \ (y : B) \ /_\varrho \vdash x^\alpha : A}$$

Gratzer, Kavvos, Nuyts, Birkedal, "Multimodal Dependent Type Theory", 2011.15021

Division is an adjoint

Recall the introduction rule of $\mu\boxdot A$:

$$\frac{\Gamma/_\mu \vdash a : A}{\Gamma \vdash \mathsf{mod}(a) : \mu\boxdot A}$$

This suggests that $(-/_\mu)$ is a left adjoint to $\mu\boxdot -$.

Theorem (∼GKNB)

MTT with mode theory $\mathcal{M}$ can be interpreted in any 2-functor $\mathscr{C} : \mathcal{M} \to \mathit{CwF}$ such that

- *Each category $\mathscr{C}_p$ models MLTT, and*
- *Each map $\mathscr{C}_\mu : \mathscr{C}_p \to \mathscr{C}_q$ is a dependent right adjoint.*

Left adjoints to modality functors

Thus, in any chain of adjoint functors, we can model all but the leftmost as modalities in MTT. Sometimes we can do even better:

Example

In a cohesive topos with $\int \dashv \flat \dashv \sharp$, we can model $\flat$ and $\sharp$ as MTT modalities. And since $\int$ is an idempotent monadic modality, we can represent it internally as a localization (RSS).

But this doesn't always work:

Example

The category of condensed*/pyknotic sets has $\flat \dashv \sharp$ but not $\int$.
It seems we can only model $\sharp$, and $\flat$ is a comonad, so not internal.

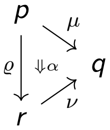
Outline

The category of liftings

Given $\mu : p \to q$ and $\nu : r \to q$, let Fact_ν^μ be the category:

- Objects are pairs (ϱ, α) with $\varrho : p \to r$ and $\alpha : \mu \Rightarrow \nu \circ \varrho$
- Morphisms $(\varrho, \alpha) \to (\varrho', \alpha')$ are $\beta : \varrho \Rightarrow \varrho'$ s.t. $(\nu \triangleleft \beta) \circ \alpha = \alpha'$.

Let $\mathscr{C} : \mathcal{M} \to \mathcal{C}at$, with $\mu : p \to q$ and $\nu : r \to q$.
For $A \in \mathscr{C}_p$, we have a functor



$$\begin{array}{rcl} \mathrm{Fact}_\nu^\mu & \to & \mathscr{C}_r \\ (\varrho, \alpha) & \mapsto & \mathscr{C}_\varrho(A) \end{array}$$

The category of liftings

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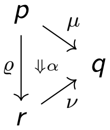

$$\begin{array}{ccc} \mathsf{Fact}^{\mu}_{\nu} & \to & \mathscr{C}_r \\ (\varrho, \alpha) & \mapsto & \mathscr{C}_{\varrho}(A) \end{array}$$

The ∼LSR approach to $(x :^{\mu} A)/\nu$ has one variable $(x^{\alpha} :^{\varrho} A)$ for each $(\varrho, \alpha) \in \mathsf{Fact}^{\mu}_{\nu}$, which semantically means the product

$$\prod_{(\varrho,\alpha) \in \mathsf{Fact}^{\mu}_{\nu}} \mathscr{C}_{\varrho}(A).$$

Obviously, this ignores the morphisms in $\mathsf{Fact}^{\mu}_{\nu}$!

Semantic context division

This suggests $(x :^{\mu} A)/\nu$ should be the limit of $(x :^{\varrho} A)$ over $\mathsf{Fact}^{\mu}_{\nu}$.

It's unclear if this makes sense syntactically in general, but we can use it semantically to define division on a context extension:

$$\big(\Gamma, (x :^{\mu} A)\big)/\nu \;\equiv\; \big(\Gamma/\nu,\; \lim_{(\varrho,\alpha)\in \mathsf{Fact}^{\mu}_{\nu}} (x :^{\varrho} A)\big)$$

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For this to make sense as a definition of $/\nu$, we need either:

1. Contexts are defined inductively, with "extension by a limit" as a constructor, and this is a recursive definition of $/\nu$.
2. Contexts are defined coinductively, with $/\nu$ as a destructor, and this is a corecursive definition of context extension!

Both should work, but the second is easier.

("Modal contextual category" vs "Modal category with families")

The co-dextrification

Given $\mathscr{C} : \mathcal{M} \to \mathcal{C}at$, let an object of $\widehat{\mathscr{C}}_r$ consist of

1. For each $\mu : p \to r$ in $\mathcal{M}$, an object $\Gamma_{/\mu} \in \mathscr{C}_p$.
2. For each $\varrho : p \to q$ and $\alpha : \mu \Rightarrow \nu \circ \varrho$, a map $\Gamma_{/\nu} \to \mathscr{C}_\varrho(\Gamma_{/\mu})$.
3. Coherence axioms.

Theorem (S.)

Let $\mathscr{C} : \mathcal{M} \to \mathcal{C}at$, where each $\mathscr{C}_p$ has, and each $\mathscr{C}_\mu$ preserves, $\mathcal{M}$-sized limits. Then $\widehat{\mathscr{C}} : \mathcal{M} \to \mathcal{C}at$, each $\widehat{\mathscr{C}}_\mu$ has a left adjoint, and the types in $\widehat{\mathscr{C}}_p$ are those of $\mathscr{C}_p$.

Thus, we can interpret MTT in $\widehat{\mathscr{C}}$ to reason about $\mathscr{C}$.

Shulman, "Semantics of multimodal adjoint type theory", 2303.02572

Outline

Negative modalities

If $\mu \dashv \mu^\dagger$ is an adjunction in $\mathcal{M}$, we can define a negative $\mu \diamond\!\!\!\rightarrow A$:

$$\frac{\Gamma/_{\mu^\dagger} \vdash M : A}{\Gamma \vdash \mathsf{mod}(M) : \mu \diamond\!\!\!\rightarrow A} \qquad \frac{\Gamma/_{\mu} \vdash M : \mu \diamond\!\!\!\rightarrow A}{\Gamma \vdash \mathsf{open}(M) : A}$$

with η-conversion, $(\mu \boxdot -) \dashv (\mu \diamond\!\!\!\rightarrow -)$, and $\mu \diamond\!\!\!\rightarrow A \simeq \mu^\dagger \boxdot A$.

- No $\mu^\dagger$-variables, e.g. $\flat \dashv \sharp$ uses only crisp ($\flat$) variables.
- Modeled in co-dextrifications with freely added right adjoints.

MTT + negatives = Multimodal Adjoint Type Theory (MATT).

Indexed modalities and interval variables

Let $\iota : p \to p$ be a modality with $\mathbf{0}, \mathbf{1} : \iota \Rightarrow 1_p$.
Its indexed modality Id_A has introduction rule

$$\frac{\Gamma/\iota \vdash M : A}{\Gamma \vdash \lambda M : \mathsf{Id}_A(M^{\mathbf{0}}, M^{\mathbf{1}})}$$

So Γ/ι acts like extension by an interval variable: $(\Gamma, i : \mathbb{I})$.

Depending on what else we put in, this acts like

- Cubical type theories
- Internal parametricity type theories
- Simplicial type theory

Is there a general theory of “indexed modalities”?

Parametric adjoints

The elimination rule for cubical path-types looks negative:

$$\frac{\Gamma \vdash M : \mathsf{Id}_A(x, y) \qquad \Gamma \vdash d : \mathbb{I}}{\Gamma \vdash M\, d : A}$$

The cubical cylinder $\Gamma \mapsto \Gamma \times \mathbb{I}$ isn't a right adjoint, but it is:

Definition

A functor $F : \mathcal{C} \to \mathcal{D}$ is a parametric right adjoint if the induced functor $\mathcal{C} \to \mathcal{D}/F1$ is a right adjoint.

To give $\nu\boxdot A$ a negative eliminator, it suffices for $/\nu$ to have a parametric left adjoint L_ν:

$$\frac{L_\nu(\Gamma, r) \vdash M : \nu\boxdot A \qquad \Gamma \vdash r : \diamond/\nu}{\Gamma \vdash \mathsf{open}_r(M) : A}$$

Gratzer, Cavallo, Kavvos, Guatto, Birkedal, "Modalities and Parametric Adjoints", ToCL 2022

Left liftings

Recall in the co-dextrification $(x :^{\mu} A)/\nu = \lim_{(\varrho,\alpha)\in\mathsf{Fact}^{\mu}_{\nu}}(x :^{\varrho} A)$.
Can we make sense of this syntactically?

Left liftings

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Can we make sense of this syntactically?

- If $\mathsf{Fact}_{\nu}^{\mu}$ has an initial object $\eta : \mu \Rightarrow \nu \circ (\mu/\nu)$, then

$$\lim_{(\varrho,\alpha)\in \mathsf{Fact}_{\nu}^{\mu}}(x :^{\varrho} A) = (x :^{\mu/\nu} A)$$

  Such a μ/ν is called a left lifting of μ along ν.

Left liftings

Recall in the co-dextrification $(x :^{\mu} A)/_{\nu} = \lim_{(\varrho,\alpha)\in \mathsf{Fact}^{\mu}_{\nu}} (x :^{\varrho} A)$. Can we make sense of this syntactically?

- If $\mathsf{Fact}^{\mu}_{\nu}$ has an initial object $\eta : \mu \Rightarrow \nu \circ (\mu/\nu)$, then

  $$\lim_{(\varrho,\alpha)\in \mathsf{Fact}^{\mu}_{\nu}} (x :^{\varrho} A) = (x :^{\mu/\nu} A)$$

  Such a μ/ν is called a left lifting of μ along ν.
- If $\mathsf{Fact}^{\mu}_{\nu}$ is a disjoint union of categories with initial objects $\eta_i : \mu \Rightarrow \nu \circ (\mu/\nu)_i$ (called a left multi-lifting), then

  $$\lim_{(\varrho,\alpha)\in \mathsf{Fact}^{\mu}_{\nu}} (x :^{\varrho} A) = \prod_i (x :^{(\mu/\nu)_i} A)$$

  In particular, if $\mathsf{Fact}^{\mu}_{\nu}$ is empty, then x vanishes in $(x :^{\mu} A)/_{\nu}$.

Many mode theories have left (multi-)liftings, and in practice we often use context divisions that compute this way.

Computing modalities

Most modalities preserve products:

$$\mu\boxdot(A \times B) \simeq \mu\boxdot A \times \mu\boxdot B$$
$$c(A \times B) \simeq c(A) \times c(B)$$
$$\mathsf{Id}_{A\times B}(u, v) \simeq \mathsf{Id}_A(\mathsf{fst}\, u, \mathsf{fst}\, v) \times \mathsf{Id}_B(\mathsf{snd}\, u, \mathsf{snd}\, v)$$

Some preserve other type-formers too:

$$c(A \to B) \simeq c(A) \to c(B)$$
$$\mathsf{Id}_{A\to B}(f, g) \simeq \textstyle\prod_{x:A} \mathsf{Id}_B(f\,x, g\,x)$$

When can we turn these into computation laws for the LHS?

- Higher Observational Type Theory* does this for Id
- Displayed Type Theory† does it for modalities $\lozenge$ and $(-)^{\mathsf{d}}$

---

* Joint WIP with Altenkirch and Kaposi

† Joint WIP with Kolomatskaia

Modal (co)inductive types

- The positive modality of μ acts like an inductive datatype with one constructor $\mathsf{mod} : (x :^{\mu} A) \to \mu\boxdot A$.
- The negative modality of $\mu \dashv \mu^{\dagger}$ acts like a record type with one destructor $\mathsf{open} : (x :^{\mu} \mu\diamond\!\!\to A) \to A$.

We can also consider more general inductive, coinductive, and record types with modal constructors and destructors.

Modal (co)inductive types

- The positive modality of μ acts like an inductive datatype with one constructor mod : $(x :^{\mu} A) \to \mu\boxdot A$.
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We can also consider more general inductive, coinductive, and record types with modal constructors and destructors.

- A higher inductive type is an inductive datatype X with constructors valued in Id_X.

We can also consider inductive and coinductive types with constructors and destructors in other modalities. (Kolomatskaia, WIP)

Modal type formers

We can modally parametrize type formers, e.g. for $p \xrightarrow{\mu} q \xrightarrow{\nu} r$

$$\Sigma_{\mu,\nu} : (A :^{\nu\circ\mu} \mathcal{U}_p) \to (B :^{\nu} (x :^{\mu} A) \to \mathcal{U}_q) \to \mathcal{U}_r$$

$$\Pi_{\mu,\nu} : (A :^{\nu\circ\mu} \mathcal{U}_p) \to (B :^{\nu} (x :^{\mu} A) \to \mathcal{U}_q) \to \mathcal{U}_r$$

$$\mathcal{U}_q \ \mathsf{type}_p \qquad\qquad X :^{\mu} \mathcal{U}_q \vdash \mathsf{El}(A) \ \mathsf{type}_q$$

and assume that they exist only for certain μ, ν, even if μ, ν are isomorphisms in $\mathcal{M}$ (as for 2LTT).

Examples

- Π-types of (∞-)categories (Neumann's talk today)
- Smooth/proper families as modes (Anel's talk yesterday)
- Pure type systems?
- Universe levels?

Homotopical models

- Any $(\infty, 1)$-topos can be presented by a model category that interprets Book HoTT.
- Any finite diagram of 1-topoi has a co-dextrification that interprets MATT.

Question

Can a finite diagram of $(\infty, 1)$-topoi be presented by a diagram of model categories interpreting MATT?

- If the $(\infty, 1)$-topoi are 1-localic, we can work with the 1-sites.
- Some other special cases are tractable.
- Do we need to let $\mathcal{M}$ be an $(\infty, 2)$-category?

Implementations

Can we implement general modal type theories?

- Gratzer: MTT satisfies normalization
- SGB: Prototype implementation of locally posetal MTT

Potential issues:

- Substitutions in MTT have no "list of terms" canonical form: generated inductively by terms, divisions, composites, etc.
- When evaluating a variable x^α in an NbE environment, we have to substitute the resulting "value" along α.
- Co-dextrification with negatives has freely added adjoints. But such 2-categories can have undecidable equality (DPP).

Gratzer, "Normalization for multimodal type theory", 2106.01414
Stassen, Gratzer, Birkedal, "mitten: a flexible multimodal proof assistant", preprint 2022
Dawson, Paré, Pronk, "Undecidability of the Free Adjoint Construction", ACS 2003

Thank you

Thanks for listening!