Homotopy Type Theory 2023
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Propositional dependent type theories: a conservativity result for homotopy elementary types.

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The notion of conservativity

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This property corresponds to the following: whenever T_2 proves a term-judgement $\Gamma \vdash a : A$, then T_1 proves a term-judgement $\Gamma \vdash \tilde{a} : A$.

When this happens, we say that T_2 is *conservative* over T_1.

Hofmann's conservativity result

If T_2 is an extensional type theory and T_1 is an intensional type theory with some (fundamental) additional extensional assumptions, e.g.

$$x : A, p : x = x \vdash p = r(x)$$

then T_2 is conservative over T_1. The proof is contained in:

Martin Hofmann, *Conservativity of equality reflection over intensional type theory*, 1996.

Intensional identity types

Formation & Introduction rules.

$$\frac{\vdash A : \textsc{Type}}{\begin{array}{c} x, x' : A \vdash x = x' : \textsc{Type} \\ x : A \vdash r(x) : x = x \end{array}}$$

Path Elimination & Computation rules.

$$\frac{\begin{array}{c} \vdash A : \textsc{Type} \\ x, x' : A;\ p : x = x' \vdash C(x, x', p) : \textsc{Type} \\ x : A \vdash q(x) : C(x, x, r(x)) \end{array}}{\begin{array}{ll} x, x' : A;\ p : x = x' \vdash & J(q, x, x', p) : C(x, x', p) \\ x : A \vdash & J(q, x, x, r(x)) \equiv q(x) \end{array}}$$

Propositional identity types

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Propositional identity types in the literature

Propositional identity types appear in:

- Marc Bezem, Thierry Coquand, Simon Huber, *A model of type theory in cubical sets*, 2014.
- Benno van den Berg, *Path categories and propositional identity types*, 2018.
- Benno van den Berg, Martijn den Besten, *Quadratic type checking for objective type theory*, 2021.

Dependent sum types

Formation and Introduction rules.

$$\frac{\vdash A : \textsc{Type} \qquad x : A \vdash B(x) : \textsc{Type}}{\vdash \Sigma_{x:A} B(x) : \textsc{Type} \qquad x : A, y : B(x) \vdash \langle x, y \rangle : \Sigma_{x:A} B(x)}$$

Elimination and Computation rules.

$$\frac{\vdash A : \textsc{Type} \qquad x : A \vdash B(x) : \textsc{Type} \qquad u : \Sigma_{x:A} B(x) \vdash C(u) : \textsc{Type} \qquad x : A;\ y : B(x) \vdash c(x, y) : C(\langle x, y \rangle)}{u : \Sigma_{x:A} B(x) \vdash \mathrm{split}(c, u) : C(u) \qquad x : A;\ y : B(x) \vdash \mathrm{split}(c, \langle x, y \rangle) \equiv c(x, y)}$$

Propositional dependent sum types

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Aim of the talk

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By adapting Hofmann's argument, we compare PTT to extensional type theories, looking for a conservativity result of such a type theory over PTT.

h-elementary types

Definition

The class of the h-elementary type-judgements is the smallest class $\mathcal{F}$ of type-judgements of PTT such that:

- atomic type-judgements $\gamma \vdash S : \textsc{Type}$ belongs to $\mathcal{F}$ if S is an h-set;
- the family $\mathcal{F}$ is closed up to applying the formation rules of $=$, Π and Σ.

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A type-judgement $\delta \vdash T(\delta) : \textsc{Type}$ of PTT is h-elementary if it belongs to this class.

Let ETT be the extensional type theory whose atomic types are the atomic h-sets of PTT.

Let $|\cdot| : \text{PTT} \rightharpoonup \text{ETT}$ be the interpretation of:

1. the h-elementary contexts of PTT,
2. the h-elementary type-judgements of PTT in h-elementary context,
3. the term-judgements of PTT in h-elementary type and context,

i.e. the h-elementary sub-theory of PTT, in ETT.

Theorem

Whenever $\gamma : \Gamma$ *is an h-elementary context of* PTT *and:*

$$\gamma \vdash A(\gamma) : \textsc{Type}$$

is an h-elementary type-judgement of PTT, *if* ETT *infers* $|\gamma| \vdash_{\texttt{ext}} a(|\gamma|) : |A(\gamma)|$ *then* PTT *infers:*

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If the only atomic types of PTT *are the empty type, the unit type, the type of booleans and the natural numbers type, then* ETT *is conservative over* PTT.

Categories with attributes...

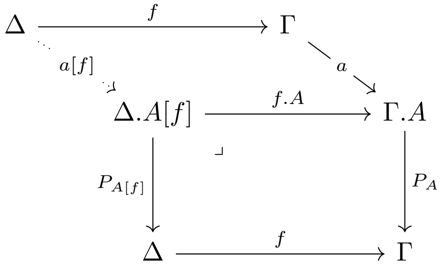
A *category with attributes* $(\mathcal{C}, \textsc{tp}, -.-, P)$ consists of:

- A category $\mathcal{C}$ of *semantic contexts* Γ, Δ, ...
- A presheaf $\mathcal{C}^{\mathrm{op}} \xrightarrow{\textsc{tp}} \textsc{Set}$ of *semantics types* A, B... in some semantic context Γ
- A *semantic context extension* $\int \textsc{tp} \to \mathcal{C}$, denoted as $(\Gamma, A) \mapsto \Gamma.A$
- A cartesian natural family of display maps $\Gamma.A \xrightarrow{P_A} \Gamma$

The *semantic terms* of A are the sections $\Gamma \xrightarrow{a} \Gamma.A$ of $\Gamma.A \xrightarrow{P_A} \Gamma$.

...i.e. models of dependent type theories

Suppose that we are given a morphism $\Delta \xrightarrow{f} \Gamma$. Then we can define the mapping $a \mapsto a[f]$ as follows:



Idea of the proof

1. One can interpret judgements of a given dependent type theory into an appropriate category with attributes.
2. One can say when a category with attributes is endowed with (semantic) extensional/intensional/propositional id types.

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2. One can say when a category with attributes is endowed with (semantic) extensional/intensional/propositional id types.
3. This provides a notion of *sound semantics* for the corresponding extensional/intensional/propositional type theory.
4. *Proof strategy.* Constructing a model M (according to this notion of semantics) of ETT, such that the interpretation of the h-elementary sub-theory of PTT in M is surjective (on contexts, types and terms).

Idea of the proof

In fact, let us assume that we found such a model M.

And let us assume that $\gamma \vdash A : \textsc{Type}$ is h-elementary in PTT and that $|\gamma| \vdash a : |A|$ in ETT.

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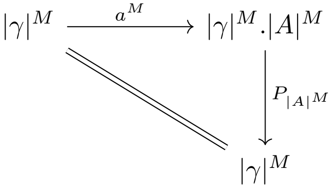

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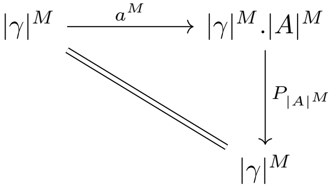

Then there is some h-elementary term-judgement $\delta \vdash b : B$ in PTT whose interpretation in M coincides with the a^M.

By some initiality argument and by "the nature of M", one can assume that $\delta \equiv \gamma$ and $B \equiv A$, therefore $\gamma \vdash b : A$.

How to built M

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We must identify paths with reflexivities:

$$x, y : A \vdash p : x = y \quad \leftrightarrow \quad x : A \vdash r(x) : x = x$$

in a way that maintains the dependent type theoretic structure.

How to identify types

We identify two types $\gamma \vdash A(\gamma) : \textsc{Type}$ and $\delta \vdash B(\delta) : \textsc{Type}$ if between them there is a canonical homotopy equivalence.

Canonical homotopy equivalences are defined inductively on the complexity of the types:

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- If $A(\gamma) \equiv B(\delta)$ is an atomic type, then the identity map is canonical.
- If $A(\gamma) \equiv x_1 =_{A'(\gamma)} x_2$ and $B(\delta) \equiv y_1 =_{B'(\delta)} y_2$ and there is canonical $f\colon A'(\gamma) \to B'(\delta)$ together with:

  $$\delta \vdash q_1 : f(x_1) =_{B'(\delta)} y_1$$
  $$\delta \vdash q_2 : f(x_2) =_{B'(\delta)} y_2$$

  then the induced equivalence $p \mapsto q_1^{-1} \bullet f(p) \bullet q_2$ between $A(\gamma)$ and $B(\delta)$ is canonical.

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  Hence, if $\vdash \overline{p} : x_1 = x_2$ then $p \mapsto \overline{p}^{-1} \bullet p \bullet r(x_2)$ is (essentially) canonical $x_1 = x_2 \to x_2 = x_2$ and identifies $\overline{p}$ with $r(x_2)$.

So what is M?

The semantic contexts of M are the equivalence classes of contexts of PTT modulo canonical context homotopy equivalences.

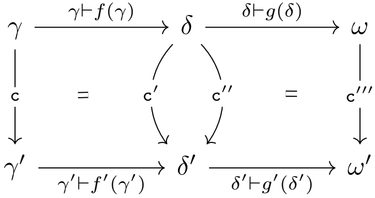
Two morphisms of contexts f and f' as in the diagram:

$$
\begin{array}{ccc}
\gamma & \xrightarrow{\;f(\gamma)\;} & \delta \\
{\scriptstyle \mathsf{c}(\delta)}\big\downarrow & /\!/ & \big\downarrow{\scriptstyle \mathsf{c}'(\delta)} \\
\gamma' & \xrightarrow[\;f'(\gamma')\;]{} & \delta'
\end{array}
$$

represent the same morphism $[\gamma] \to [\delta]$ in M.

Issues & why h-elementary types

- If $[f] = [f']$ and $[g] = [g']$, that is:



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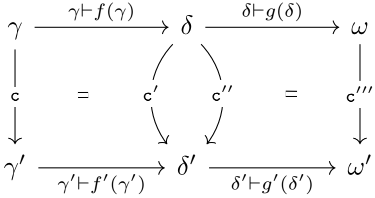

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Hence we need to allow two identity types to be identified only when they are mere propositions. We restrict ourselves to types with *h-propositional identities* and contexts with *h-propositional identities*.

Issues & why h-elementary types

- Even in this restriction, a naturality square:

$$\begin{array}{ccc} \Delta.A[f] & \xrightarrow{f.A} & \Gamma.A \\ {\scriptstyle P_{A[f]}}\big\downarrow & & \big\downarrow{\scriptstyle P_A} \\ \Delta & \xrightarrow{\quad f \quad} & \Gamma \end{array}$$

  is a weak pullback but not necessarily a pullback.

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  is a weak pullback but not necessarily a pullback.

  Hence we need to make a further restriction, i.e. we only allow types to be identified if they are h-elementary and in h-elementary context.

  In this case we actually obtain a category with attributes M that happens to be a model of ETT.

In order to have a look at the argument in detail:

Matteo Spadetto, *A conservativity result for homotopy elementary types in dependent type theory*, 2023.
arXiv 2303.05623