Groupoidal realizability over cubical λ-calculus

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Motivation

Realizability is a way to make precise the BHK interpretation: realizers witness the truth of propositions.

In the homotopy interpretation, paths are witnesses of identifications.

Thus, in the setting of ITT/HoTT, one might expect realizers themselves to carry higher-dimensional structure.

This is so in *computational higher-dimensional type theory* [Angiuli-Harper-Wilson '17]: realizers are terms of a cubical programming language.

Landscape of realizability models

$$
\begin{array}{ccccc}
\mathbf{PAsm}(\mathcal{A}) & \hookrightarrow & \mathbf{Asm}(\mathcal{A}) & \hookrightarrow & \mathbf{RT}(\mathcal{A}) \\
\uparrow & \lrcorner & \uparrow & & \\
\mathbf{PMod}(\mathcal{A}) & \hookrightarrow & \mathbf{Mod}(\mathcal{A}) & &
\end{array}
$$

Realizer categories

To begin with, instead of PCAs, we work with "realizer categories".
A realizer category $\mathbb{C}$ is a CCC with:

- an interval $\mathbb{I} \in \mathbb{C}$ *qua* co-groupoid (à la [Warren '12]),
- a universal object $U \in \mathbb{C}$ (every object in $\mathbb{C}$ is a retract of U up to homotopy).

The interval supplies a notion of homotopy in $\mathbb{C}$:

$$\begin{array}{ccccc} A \times 1 & \xrightarrow{A\times 0} & A \times \mathbb{I}_1 & \xleftarrow{A\times 1} & A \times 1 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle \pi_1} \\ A & \xrightarrow[f]{} & B & \xleftarrow[g]{} & A \end{array}$$

as well as a fundamental groupoid construction:

$$\Pi := (-)^{\mathbb{I}} : \mathbb{C} \to \mathbf{Gpd}$$

$\mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U)$

A partitioned groupoidal assembly X consists of a groupoid X and a functor:

$$\| - \|_X : X \to \Pi U$$

X is modest when $\|-\|_X$ is fully faithful.

A morphism $X \to Y$ of partitioned groupoidal assemblies is a functor $F : X \to Y$ such that there exists a map $e : U \to U$ in $\mathbb{C}$ and a natural iso:

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ {\scriptstyle \|-\|_X}\big\downarrow & \overset{\epsilon}{\nearrow} & \big\downarrow{\scriptstyle \|-\|_Y} \\ \Pi U & \xrightarrow[\Pi(e)]{} & \Pi U \end{array}$$

$\mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U)$

Proposition

$\mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U)$ *is weakly cartesian closed.*

$\mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U)$ possesses an interval.

Taking 2-cells to be homotopies wrt this interval endows $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U)$ with the structure of a (2,1)-category.

Proposition

If $(\mathbb{C}, \mathbb{I})$ *is finitely complete as a (2,1)-category then so is* $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U)$.

$\mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U)$

Theorem

If $(\mathbb{C}, \mathbb{I})$ is finitely complete, then $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U)$ is a model of 1-truncated type theory with:

- *weak Π-types (no η / function extensionality),*
- *an impredicative universe of 1-types (given by the modest fibrations).*

In particular, $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U)$ is a path category [van den Berg - Moerdijk '18] with:

- fibrations: maps in $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U)$ whose underlying functor is a Grothendieck fibration,
- weak equivalences: equivalences in the (2,1)-category $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U)$.

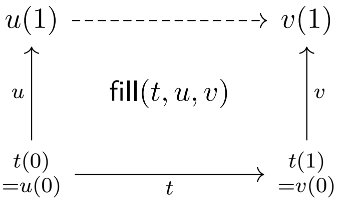
Path objects are given by exponentiation with the interval.

Cubical λ-calculus

$$\frac{\Gamma, x \mid \Psi \vdash t}{\Gamma \mid \Psi \vdash \lambda x.t} \qquad \frac{\Gamma \mid \Psi \vdash t \qquad \Gamma \mid \Psi \vdash u}{\Gamma \mid \Psi \vdash \mathsf{app}(t, u)}$$

$$\frac{\Gamma \mid \Psi, i \vdash t}{\Gamma \mid \Psi \vdash \langle i \rangle t} \qquad \frac{\Gamma \mid \Psi \vdash t \qquad \varepsilon = 0, 1}{\Gamma \mid \Psi \vdash t@\varepsilon}$$

$$\frac{\begin{array}{c}\Gamma \mid i \vdash t \qquad \Gamma \mid j \vdash u, v \\ \Gamma \mid \cdot \vdash t[0/i] = u[0/j] \\ \Gamma \mid \cdot \vdash t[1/i] = v[0/j]\end{array}}{\Gamma \mid i, j \vdash \mathsf{fill}(t, u, v)}$$



Cubical λ-calculus

The (regular) untyped λ-calculus gives rise to a CCC with universal object [Scott '80].

One takes the idempotent completion of the monoid of λ-terms satisfying $t = \lambda x.tx$ (with composition $t \circ u := \lambda x.t(u(x))$ and unit $\lambda x.x$).

Is there a similar construction for cubical λ-calculus (giving rise to a realizer category)?

Realizability over cubical λ-calculus

Construct a groupoid out of the cubical λ-calculus:

- Objects: closed terms in empty dimension context:

$$\cdot \mid \cdot \vdash t$$

- Morphisms: equivalence classes of closed terms in singleton dimension context:

$$[\ \cdot \mid i \vdash \alpha\ ] : \alpha(0) \to \alpha(1)$$

  Two such terms are identified up to homotopy rel. endpoints.

Use this groupoid in the place of ΠU. Realizers of functors are pairs consisting of a term $\cdot \mid \cdot \vdash e$ and a natural iso.

Future work

Finish work on groupoidal realizability over cubical λ-calculus.

Investigate regular and exact completions (in a suitable, higher-dimensional sense) of partitioned groupoidal assemblies.

- Is the exact completion a topos?
- Does the regular completion contain an impredicative universe? If so, is it univalent? What about propositional resizing?

Weak ∞-groupoidal realizability.

Thanks!