Cofibrancy of The Exo-type of Natural Numbers

Elif Uskuplu

University of Southern California

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References

In our research, we have made some novel contributions to the existing literature on 2LTT and its applications. The primary sources that we used are:

1. [ACKS19] Danil Annenkov, Paolo Capriotti, Nicolai Kraus, and Christian Sattler, *Two-level type theory and applications*, Available on arXiv:1705.03307v3, 2019.

2. [ANST21] Benedikt Ahrens, Paige Randall North, Michael Shulman, and Dimitris Tsementzis. *The univalence principle*, Available on arXiv:2102.06275, Feb 2021.

Outline

Two-Level Type Theory (2LTT)

2LTT [ACKS19] consists of two layers:

- Bottom – usual HoTT (objects are called *types*, e.g. $\mathbf{0}$, $\mathbf{1}$, $\mathbb{N}$, $\sum$, $\prod$, $=$, $\times$, $+$)
- Top – MLTT with UIP (objects are called *exo-types*, e.g. $\mathbf{0}^e$, $\mathbf{1}^e$, $\mathbb{N}^e$, $\sum^e$, $\prod^e$, $=^{\mathsf{e}}$, $\times^e$, $+^e$)

Type formers are defined in both levels similarly.

We assume each type is an exo-type, but not vice-versa. In [ACKS19], it is assumed a coercion from types to exotypes, but we follow the convention in [ANST21], and assume the coercion is the inclusion.

The top layer may be understood as the internalised meta-theory of the bottom.

Fibrant Exo-types

We call an exo-type A fibrant if it is isomorphic (w.r.t. exo-equality) to a type B.

Properties. ([ACKS19], Lemma 3.5)

- The unit exo-type is fibrant.
- If $A : \mathcal{U}^e$ is a fibrant exo-type and $B : A \to \mathcal{U}^e$ is a family of fibrant exo-types, then both $\sum^e_{a:A} B(a)$ and $\prod^e_{a:A} B(a)$ are fibrant.
- Fibrancy may not be preserved under $+^e$, and $\mathbf{0}^e$ and $\mathbb{N}^e$ may not be fibrant, but these statements can be added as axioms.

Cofibrant Exo-types

We call an exo-type $A : \mathcal{U}^e$ *cofibrant* if for any family of **types** $Y : A \to \mathcal{U}$,

i. the exo-type $\prod^e_{a:A} Y(a)$ is fibrant, and
ii. if each $Y(a)$ is contractible, the fibrant match of $\prod^e_{a:A} Y(a)$ is contractible.

Properties. ([ACKS19], Lemma 3.25)

- Any fibrant exo-type is cofibrant.
- The exo-empty type is cofibrant. If $C, D : \mathcal{U}^e$ are cofibrant, then so are $C \times^e D$ and $C +^e D$.
- If $A : \mathcal{U}^e$ is a cofibrant exo-type and $B : A \to \mathcal{U}^e$ is a family of cofibrant exo-types, then $\sum^e_{a:A} B(a)$ is cofibrant.

Cofibrant Exo-types

Theorem. The second condition of the cofibrancy definition for A is equivalent to the following:

(Funext for cofibrant types). For any $f, g : \prod_{a:A}^{e} Y(a)$, if $f(a) = g(a)$ for each $a : A$, then $r(f) = r(g)$ where $FM : \mathcal{U}$ and $r : \prod_{a:A}^{e} Y(a) \to FM$ is an isomorphism.

In our current context, the equivalence bears resemblance to one found in standard HoTT: `funext` is true if and only if the dependent function types of any contractible family are also contractible. The proof in our case follows a similar structure, but it requires additional attention to distinct equalities.

Outline

Is $\mathbb{N}^e$ cofibrant?

Note that since we do not assume elimination from a type to an exo-type, we cannot define, for example, a map $\mathbb{N} \to \mathbb{N}^e$ by induction.

It does not seem to be possible to prove that $\mathbb{N}^e$ is cofibrant, but this is a reasonable axiom to add since it holds in some models.

We studied some of the consequences of this axiom and tried to obtain a general class of models of 2LTT where the axiom holds.

Why do we care?

One of the original motivations for 2LTT was to define semisimplicial types, but there was a problem defining the type of untruncated semisimplicial types.

Voevodsky's solution was to assume that exo-nat is fibrant, which works for simplicial sets but may not hold in all infinity-toposes.

But assuming cofibrancy of exo-nat also allows for defining a fibrant type of untruncated semisimplicial types with a wider syntax, including models for all infinity-toposes (Example 2 below).

Assume $\mathbb{N}^e$ is cofibrant

List exo-types

For an $A : \mathcal{U}^e$ we define $\mathsf{List}^e(A) : \mathcal{U}^e$ of **finite exo-lists** of terms of A, which has constructors

- $[]^e : \mathsf{List}^e(A)$
- $::^e : A \to \mathsf{List}^e(A) \to \mathsf{List}^e(A)$

It is easy to see that

$$\mathsf{List}^e(A) \cong \sum_{n:\mathbb{N}^e}^{e} A^n.$$

Theorem. When A is cofibrant, thanks to this isomorphism, $\mathsf{List}^e(A)$ is cofibrant. Conversely, if List^e preserves cofibrancy, then $\mathbb{N}^e$ is cofibrant.

Assume $\mathbb{N}^e$ is cofibrant

Exo Types of Binary Trees

For $N, L : \mathcal{U}^e$ we define $\mathsf{BinTree}^e(N, L) : \mathcal{U}^e$ of **binary exo-trees** with node values of N and leaf values of L, which has constructors

- $\mathsf{leaf}^e : L \to \mathsf{BinTree}^e(N, L)$
- $\mathsf{node}^e : \mathsf{BinTree}^e(N, L) \to N \to \mathsf{BinTree}^e(N, L) \to \mathsf{BinTree}^e(N, L)$

Note that we can define the exo-type of unlabeled binary trees and obtain

$$\mathsf{BinTree}^e(N, L) \cong \sum\nolimits^e_{t:\mathsf{UnLBinTree}^e} \left(N^{\#\text{ of nodes of } t} \times^e L^{\#\text{ of leaves of } t} \right).$$

Assume $\mathbb{N}^e$ is cofibrant

Exo Types of Binary Trees

Theorem.
It is known that there is a one-to-one correspondence between unlabeled binary trees and balanced parenthesizations.

The second one is obtained by a dependent sum on the list type of parentheses, and this sum exo-type is cofibrant. This yields that the exo-type of unlabeled binary trees, and hence the exo-type of binary trees, is cofibrant.

Syntax & Formalization

We also formalized all these results about cofibrancy and more in Agda[1]. We used one of the new features of Agda that enable a sort *SSet* for exo-types.

Our work on this subject is a pioneering study regarding Agda's new feature. Based on the data we obtained from this, we also conducted a documentation study on 2LTT. One can read the details of this feature in the documentation[2].

[1] `https://github.com/UnivalencePrinciple/2LTT-Agda`

[2] `https://agda.readthedocs.io/en/v2.6.3/language/two-level.html`

Outline

One-Level CwFs

Well-known models of 2LTT can be found in CwFs.

A *category with families*[3] (CwF) consists of the following:

- A category of contexts $\mathcal{C}$ with a terminal object $1_{\mathcal{C}} : \mathcal{C}$.
- A presheaf $\mathtt{Ty} : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$. If $A : \mathtt{Ty}(\Gamma)$, then we say *A is a type over* Γ.
- A presheaf $\mathtt{Tm} : (\int \mathtt{Ty})^{\mathrm{op}} \to \mathsf{Set}$. If $a : \mathtt{Tm}(\Gamma, A)$, then we say *$a$ is a term of A*.
- For any $\Gamma : \mathcal{C}$ and $A : \mathtt{Ty}(\Gamma)$, there is an object $\Gamma.A : \mathcal{C}$ with a certain universal property. This operation is called the *context extension*.

[3] Peter Dybjer. *Internal type theory*, 1995.

Two-Level CwFs

A *two-level CwF* is a combination of two CwF structures on the same category, namely, we have $(\mathcal{C}, \mathtt{Ty}^e, \mathtt{Tm}^e, \mathtt{Ty}^f, \mathtt{Tm}^f)$ where both $(\mathcal{C}, \mathtt{Ty}^e, \mathtt{Tm}^e)$ and $(\mathcal{C}, \mathtt{Ty}^f, \mathtt{Tm}^f)$ are CwFs, and there is a natural transformation $c : \mathtt{Ty}^f \to \mathtt{Ty}^e$.

When $(\mathcal{C}, \mathtt{Ty}^e, \mathtt{Tm}^e)$ models MLTT with UIP and $(\mathcal{C}, \mathtt{Ty}^f, \mathtt{Tm}^f)$ models HoTT, the corresponding two-level CwF models 2LTT [ACKS19].

Cofibrancy in CwF Model

Roughly speaking, in a two-level CwF, cofibrancy is defined as follows:

An exo-type $A : \mathtt{Ty}^e(\Gamma)$ is called *cofibrant* if for any context Δ, morphism $\sigma : \Delta \to \Gamma$, and family of types $Y : \mathtt{Ty}^f(\Delta.A[\sigma])$, the exo-type

$$\prod\nolimits^e_\Delta (A[\sigma], c(Y)) : \mathtt{Ty}^e(\Delta)$$

is fibrant; naturally in Δ; and if $Y : \mathtt{Ty}^f(\Delta.A[\sigma])$ is contractible, then so is the fibrant match of $\prod^e_\Delta(A[\sigma], c(Y))$, which is again natural in Δ.

Example 1 [ANST21]

Let $\mathcal{C} = \mathrm{SSet}$ be the category of simplicial sets. As a presheaf category, it has a CwF structure $(\mathtt{Ty}^e, \mathtt{Tm}^e)$ like any presheaf category. Define $\mathtt{Ty}^f(\Gamma)$ be the subset of $\mathtt{Ty}^e(\Gamma)$ consisting of those types A such that the display map $\Gamma.A \to \Gamma$ is a Kan fibration, and c as the inclusion.

In this model, $\mathbb{N}^e$ is given by the external set $\mathbf{N}$ with the discrete simplicial structure. Since $\Gamma.\mathbb{N}^e \to \Gamma$ is always a Kan fibration, we have $\mathbb{N}^e$ is fibrant, and hence cofibrant.

Example 2.

Let $\mathcal{C}$ be a good model category[4]. Define $\mathtt{Ty}^e(\Gamma)$ as the set of all morphisms over Γ and $\mathtt{Ty}^f(\Gamma)$ as the set of fibrations over Γ. Define for $A : \mathtt{Ty}^e(\Gamma)$ the set $\mathtt{Tm}^e(\Gamma, P)$ as the hom-set $\mathcal{C}/\Gamma[\Gamma, \Gamma.A]$ and $\mathtt{Tm}^f$ similarly. Then we get a two-level CwF with the conversion $c : \mathtt{Ty}^f \to \mathtt{Ty}^e$ as being inclusion.

In this model $\mathbb{N}^e$ is given by the countable coproduct $\coprod_{\mathbf{N}} 1$ of copies of the terminal object. This is not fibrant for an arbitrary good model category. But we have:

Theorem. In this model, $\mathbb{N}^e$ is cofibrant.

[4]Peter LeFanu Lumsdaine, Mike Shulman, Semantics of Higher Inductive Types, 2017, arXiv:1705.07088

Example 2.

Proof Idea. Let $Y : \mathtt{Ty}^f(\Gamma.\mathbb{N}^e)$, namely, we have $Y_a : \mathtt{Ty}^f(\Gamma)$ for each $a : \mathbf{N}$. Since fibrations are closed under countable product, we can take the categorical products

$$\left(\prod_{a:\mathbf{N}^e} Y_a\right) : \mathtt{Ty}^f(\Gamma)$$

as the fibrant match of $\prod^e_\Gamma(\mathbb{N}^e, Y) : \mathtt{Ty}^e(\Gamma)$. The contractibility condition holds by a standard lemma about model categories.

General class of two-level CwFs (Ongoing study)

We say a CwF $(\mathcal{C}, \mathtt{Ty}, \mathtt{Tm})$ ***has exo-nat products*** if for any family of types $Y_a : \mathtt{Ty}(\Gamma)$ indexed by $a : \mathbf{N}$,

i. there is a type $B : \mathtt{Ty}(\Gamma)$ such that the set $\mathtt{Tm}(\Gamma, B)$ is isomorphic to the categorical product $\prod_{a:\mathbf{N}} \mathtt{Tm}(\Gamma, Y_a)$, naturally in Γ, and
ii. if also $d, c : \prod_{a:\mathbf{N}} \mathtt{Tm}(\Gamma, Y_a)$ are such that $d_a = c_a$ as terms of $Y_a : \mathtt{Ty}(\Gamma)$, then $d = c$ as terms of B, naturally in Γ.

Theorem. If $(\mathcal{C}, \mathtt{Ty}, \mathtt{Tm})$ has exo-nat products, then the presheaf two-level CwF $(\widehat{\mathcal{C}}, \widehat{\mathtt{Ty}}, \widehat{\mathtt{Tm}}, \mathtt{Ty}^f, \mathtt{Tm}^f)$ has cofibrant exo-nat.

Outline

Generalization to W-Types

Additional analysis is required for W-types to determine the necessary conditions on A and B for establishing the cofibrancy of $W_{a:A}B(a)$. As $\mathbb{N}^e$ is only one instance of W-types, it is unlikely that cofibrancy will be preserved by W-types in all cases. Hence, a general axiom can be proposed, and its semantics can also be examined.

Thanks!