Internal sums for synthetic fibered $(\infty, 1)$-categories

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From geometric morphisms. . .

- A *geometric morphism* $f : \mathcal{F} \to \mathcal{E}$ between toposes is an adjunction $f^* : \mathcal{F} \leftrightarrows E : f_*$ where f^* preserves finite limits.
- We think of $\mathcal{F}$ as a topos *over* $\mathcal{E}$ (Grothendieck's relative point of view).
- Fibered view (Bénabou, Moens, Jibladze): this corresponds to a fibration $p : \mathcal{X} \to \mathcal{E}$ of toposes, where $p^{-1}(1) \simeq \mathcal{F}$
- Question: When is a functor $F : \mathcal{E} \to \mathcal{F}$ the inverse image part of a geometric morphism?
- Answer (Bénabou '74): If and only if F preserves 1 and the *Artin gluing* $\mathrm{gl}(F) : \mathcal{F} \downarrow F \to \mathcal{E}$ is a fibration of toposes *with internal sums*

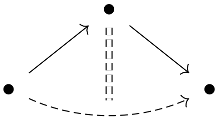
$$\begin{tikzcd}
\mathcal{F} \downarrow F \arrow[r] \arrow[d, "\mathrm{gl}(F)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{F}^{\to} \arrow[d, "\mathrm{cod}"] \\
\mathcal{E} \arrow[r, "F"'] & \mathcal{F}
\end{tikzcd}$$

. . . to finite limit fibrations

- Hence, more generally: Characterize those fibered categories $p : E \to B$ that are of the form $\mathrm{gl}(F)$ for some finite limit preserving functor $F : B \to C$ between finitely complete categories B and C.
- Moens '82: Characterization as *lextensive* fibrations. Recall: A bicomplete category $\mathcal{C}$ is *lextensive* if $\prod_{i \in I} \mathcal{C}/a_i \simeq \mathcal{C}/\coprod_{i \in I} a_i$ for all (small) families $(a_i)_{i \in I}$ of objects in $\mathcal{C}$.
- Our result: Moens' Theorem for $(\infty, 1)$-categories. . .
- . . . internally to an ∞-topos (cf. the talks by Mathieu and Louis)
- . . . in type theory!
- Generalizing Streicher's account [Str22].

The concept of $(\infty, 1)$-category

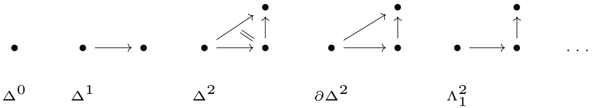
- $(\infty, 1)$**-categories:** weak composition of 1-morphisms given uniquely *up to contractibility*



- How to express this in HoTT?
- *Problem:* We have path types $(a =_A b)$, but what about directed hom types $(a \to_A b)$?
- Several possible approaches; see e.g. the talks by Matthew, Robert, Jacob, Julian, and Christopher

Simplicial HoTT

- Riehl–Shulman' 17: simplicial extension of HoTT
- add strict shapes



- add extension types (Lumsdaine–Shulman, cf. Cubical Type Theory):

Input:

- shape inclusion $\Phi \hookrightarrow \Psi$
- family $P : \Psi \to \mathcal{U}$
- partial section $a : \Pi_{t:\Phi} P(t)$

$$\begin{tikzcd}
& \widetilde{P} \arrow[d, two heads] \\
\Phi \arrow[ur, dotted, "a"] \arrow[r, hook] & \Psi \arrow[u, dotted, bend right, "b"']
\end{tikzcd}$$

$\rightsquigarrow$

Extension type $\left\langle \prod_\Psi P \middle|^{\Phi}_{a} \right\rangle$

with terms $b : \Pi_\Psi P$ such that $b|_\Phi \equiv a$.

Semantically:

$$\begin{tikzcd}
\left\langle \prod_\Psi P \middle|^{\Phi}_{a} \right\rangle \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{P}^\Psi \arrow[d, two heads] \\
\mathbf{1} \arrow[r, "a"] & \widetilde{P}^\Phi
\end{tikzcd}$$

Synthetic $(\infty, 1)$-category theory

- Enables $(\infty, 1)$-category theory in (simplicial) HoTT
- *Could ∞-Category Theory be taught to Undergraduates?* (E. Riehl, Notices of the AMS **70**(5). May 2023, 727–736)
- Internalized (parts of) the fibrational theory of Riehl–Verity's ∞-cosmoses [RV22] in [BW23] and my thesis
- Prototype proof assistant `rzk` developed by Kudasov

Extension types

- Semantically, get algebraic structure making extension types strictly stable under substitution, cf. Steve's talk; see W' 22: `arXiv:2203.07194`.
- Extension types are homotopically well-behaved, assuming a function extensionality axiom.
- De-/strictification [BW23]: $\left\langle \prod_{x:(I|\psi)} A(x) \middle|^{\varphi}_{a} \right\rangle \simeq \sum\limits_{f:\prod_{x:(I|\psi)} A(x)} \prod\limits_{x:(I|\varphi)} (a\,x = f\,x)$
- A square

$$\begin{tikzcd}
\Phi \arrow[r, "\kappa"] \arrow[d, hook, "j"] & \widetilde{P} \arrow[d, two heads, "\pi"] \\
\Psi \arrow[r, "\sigma"] \arrow[ur, dashed, "\exists!"] & B
\end{tikzcd}$$

possesses a diagonal filler uniquely up to contractibility if and only if the following proposition holds:

$$\mathrm{isContr}\left(\left\langle \Pi_{t:\Psi} P(\sigma(s)) \middle|^{\Phi}_{\kappa} \right\rangle\right)$$

Hom types I

Definition (Hom types, [RS17])

Let B be a type. Fix terms $a, b : B$. The type of *arrows in* B *from* a *to* b is the extension type

$$\mathrm{hom}_B(a,b) :\equiv (a \to_B b) :\equiv \left\langle \Delta^1 \to B \middle|^{\partial\Delta^1}_{[a,b]} \right\rangle.$$

Definition (Dependent hom types, [RS17])

Let $P : B \to \mathcal{U}$ be family. Fix an arrow $u : \mathrm{hom}_B(a,b)$ in B and points $d : P\,a$, $e : P\,b$ in the fibers. The type of *dependent arrows in* P *over* u *from* d *to* e is the extension type

$$\mathrm{dhom}_{P,u}(d,e) :\equiv (d \to^P_u e) :\equiv \left\langle \prod_{t:\Delta^1} P(u(t)) \middle|^{\partial\Delta^1}_{[d,e]} \right\rangle.$$

Hom types II

We will also be considering types of 2-cells: For arrows u, v, w in B with f, g, h in P lying above, with appropriate co-/domains, let

$$\hom_B^2(u, v; w) :\equiv \left\langle \Delta^2 \to B \Big|^{\partial\Delta^2}_{[u,v,w]} \right\rangle, \quad \mathrm{dhom}_\sigma^{2,P}(f, g; h) :\equiv \left\langle \prod_{\langle t,s\rangle:\Delta^2} P(\sigma(t,s)) \Big|^{\partial\Delta^2}_{[f,g,h]} \right\rangle.$$

$$\begin{tikzcd}
\widetilde{P} \arrow[dddd, two heads] & & e' \arrow[dr, "g"] \arrow[d, Rightarrow, "\tau"] & \\
& e \arrow[ur, "f"] \arrow[rr, "h"'] & {} & e'' \\
& & & \\
& & b' \arrow[dr, "v"] \arrow[d, Rightarrow, "\sigma"] & \\
B & b \arrow[ur, "u"] \arrow[rr, "w"'] & {} & b''
\end{tikzcd}$$

Segal, Rezk, and discrete(=groupoidal) types

Can now define synthetic $(\infty, 1)$-categories using shapes and extension types; recall Julie's talk

Definition (Synthetic $(\infty, 1)$-categories, [RS17])

- **Synthetic pre-$(\infty, 1)$-category *aka* Segal type:** types A with *weak composition*, *i.e.*:

$$\iota : \Lambda^2_1 \hookrightarrow \Delta^2 \rightsquigarrow A^\iota : A^{\Delta^2} \xrightarrow{\simeq} A^{\Lambda^2_1} \qquad \text{(Joyal)}.$$

- **Synthetic $(\infty, 1)$-category *aka* Rezk type:** Segal types A satisfying *Rezk completeness/local univalence*, *i.e.*

$$\mathrm{idtoiso}_A : \Pi_{x,y:A}(x =_A y) \xrightarrow{\simeq} \mathrm{iso}_A(x, y).$$

- **Synthetic ∞-groupoid *aka* discrete type:** types A such that *every arrow is invertible*, *i.e.*

$$\mathrm{idtoarr}_A : \Pi_{x,y:A}(x =_A y) \xrightarrow{\simeq} \hom_A(x, y).$$

Cocartesian families: Motivation

- Any type family $P : B \to \mathcal{U}$ **transforms covariantly** in paths:

$$u : a =_B b \quad \rightsquigarrow \quad u_! : P\,a \to P\,b$$

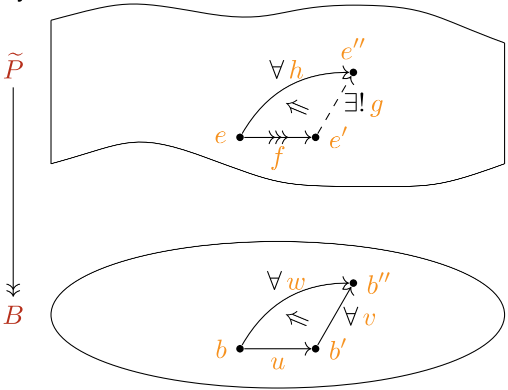
- What about the **directed** analogue? We'd like:

$$u : a \to_B b \quad \rightsquigarrow \quad u_! : P\,a \to P\,b$$

- This is true for **functorial** families $P : B \to \mathcal{U}$.
- Different notions of **fibrations**, investigated by Riehl–Shulman, and later Buchholtz–W [BW23] and W.
- Satisfy **directed arrow induction** *aka* **type-theoretic Yoneda Lemmas** (originally due to [RS17], also in [RV22]).
- Formalization (in progress) in `rzk`: `https://github.com/emilyriehl/yoneda`

Cocartesian arrows: Intuition

Intuitively: An arrow $f : e \to^P_u e'$ over $u : b \to_B b'$ is *cocartesian* if it satisfies the following universal property:



$\exists!\, \tau : f, g \Rightarrow_\sigma h$

$\forall\, \sigma : u, v \Rightarrow w$

Cocartesian arrows: Definition

Definition (Cocartesian arrows ([BW23], cf. [RV22]))

Let B be a type and $P : B \to \mathcal{U}$ be an inner family. Let $b, b' : B$, $u : \hom_B(b, b')$, and $e : P\,b$, $e' : P\,b'$. An arrow $f : \hom_{P\,u}(e, e')$ is a *(P-)cocartesian morphism* or *(P-)cocartesian arrow* iff

$$\mathrm{isCocartArr}_P\, f :\equiv \prod_{\sigma:\left\langle \Delta^2 \to B \middle|^{\Delta^1_0}_{u} \right\rangle} \; \prod_{h:\prod_{t:\Delta^1} P\,\sigma(t,t)} \mathrm{isContr}\left(\left\langle \prod_{\langle t,s\rangle:\Delta^2} P\sigma(t,s) \middle|^{\Lambda^2_0}_{[f,h]} \right\rangle \right).$$

Notice that being a cocartesian arrow is a proposition. Over a Segal base, this amounts to:

$$\mathrm{isCocartArr}_P\, f \simeq \prod_{b'':B} \; \prod_{v:\hom_B(b',b'')} \; \prod_{w:\hom_B(b,b'')} \; \prod_{\sigma:\hom^2_B(u,v;w)} \; \prod_{e'':P\,b''} \; \prod_{h:\mathrm{dhom}_{P\,w}(e,e'')}$$

$$\mathrm{isContr}\left(\sum_{g:\mathrm{dhom}_{P\,v}(e',e'')} \mathrm{dhom}^2_{P\,\sigma}(f,g;h) \right)$$

Cocartesian families: Definition

Definition (Cocartesian family ([BW23], cf. [RV22]))

Let B be a Rezk type and $P : B \to \mathcal{U}$ be a family such that $\widetilde{P}$ is a Rezk type. Then P is a *cocartesian family* if:

$$\mathrm{hasCocartLifts}\, P :\equiv \prod_{b,b':B} \prod_{u:b\to b'} \prod_{e:P\,b} \sum_{e':P\,b'} \sum_{f:e\to_u e'} \mathrm{isCocartArr}_P\, f$$

A map $\pi : E \twoheadrightarrow B$ is a *cocartesian fibration* iff $P :\equiv \mathrm{St}_B(\pi)$ is a cocartesian family.

$$\begin{tikzcd} E \arrow[d, two heads, "\pi"'] & \forall\, e \arrow[r, dashed, "\exists(!)\ \pi_!(u,e)"] & u_!^P\, e \\ B & a \arrow[r, "\forall\, u"] & b \end{tikzcd}$$

$$\rightsquigarrow (-)_!^P : \prod_{a,b:B} (a \to_B b) \to P(a) \to P(b)$$

Cocartesian families: Functoriality

- Hence, any $u : a \to_B b$ induces a functor $u_! : P\,a \to P\,b$ acting on arrows as follows:

$$\begin{tikzcd}
E \arrow[dd, two heads] & e \arrow[r, "P_!(u{,}e)"] \arrow[d, "g"'] & u_!\,e \arrow[d, dashed, "u_!\,g"] \\
 & e' \arrow[r, "P_!(u{,}e')"'] & u_!\,e' \\
B & a \arrow[r, "u"] & b
\end{tikzcd}$$

- Externally, this corresponds to a Cat-valued ∞-functor $B \to \mathrm{Cat}$, where Cat is the $(\infty, 1)$-category of small $(\infty, 1)$-categories.

Cocartesian families: Examples

1. For $g : C \to A \leftarrow B : f$, the comma projection $\partial_C : f \downarrow g \twoheadrightarrow C$.[1] (Hence, in particular the codomain projections $\partial_1 : A^{\Delta^1} \twoheadrightarrow A$.)
2. The *domain projection* $\partial_0 : A^{\Delta^1} \twoheadrightarrow A$, provided A has all pushouts.
3. For any map $\pi : E \to B$ between Rezk types, the *free cocartesian fibration*:

$$\begin{tikzcd}
\pi \downarrow B \arrow[r] \arrow[d] \arrow[dd, bend right, two heads, "L(\pi) :\equiv \partial_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "\pi"] \\
B^{\Delta^1} \arrow[r, "\partial_0"'] \arrow[d, "\partial_1"] & B \\
B &
\end{tikzcd}$$

In particular, the desired UMP holds: $- \circ \iota : \mathrm{CocartFun}_B(L(\pi), \xi) \xrightarrow{\simeq} \mathrm{Fun}_B(\pi, \xi)$ for any cocartesian fibration $\xi : F \to B$.

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[1] $f \downarrow g \simeq \Sigma_{b:B, c:C}(f\, b \to_A g\, c)$

Cocartesian families: Characterization

Theorem (Cocartesian families via transport ([BW23], cf. [RV22]))

A given family $P : B \to \mathcal{U}$ *is cocartesian if and only if the map*

$$\iota :\equiv \iota_P : E \to \pi \downarrow B, \quad \iota\,\langle b, e\rangle :\equiv \langle \mathrm{id}_b, e\rangle$$

has a fibered left adjoint $\tau :\equiv \tau_P : \pi \downarrow B \to E$ *as indicated in the diagram:*

$$\begin{tikzcd}
E \arrow[rr, bend right, "\iota"'] \arrow[rr, phantom, "\perp"] \arrow[dr, two heads, "\pi"'] & & \pi \downarrow B \arrow[ll, bend right, "\tau"'] \arrow[dl, two heads, "\partial_1"] \\
& B &
\end{tikzcd}$$

The idea is that $\tau : \pi \downarrow B \to_B E$ is the *transport map* $\tau(u, e) \equiv u_!\, e : P(u(1))$.

Cartesian and bicartesian families

- By (manual) dualization: obtain a theory of *cartesian* families $P : B \to \mathcal{U}$, with contravariant transport $u^* : P\,b \to P\,a$ and RARI condition.
- Combining both variances leads to *bicartesian* families, where $u_! \dashv u^* : P\,b \to P\,a$.

Bicartesian families: Examples

- The *Artin gluing* (or simply *gluing*) $\mathrm{gl}(F) : C \downarrow F \twoheadrightarrow B$ of a functor $F : B \to C$, for Rezk types B and C, with C having all pullbacks:

$$\begin{tikzcd}
C \downarrow F \arrow[r] \arrow[d, two heads, "\mathrm{gl}(F)"'] \arrow[dr, phantom, "\lrcorner", very near start] & C^{\Delta^1} \arrow[d, two heads, "\partial_1"] \\
B \arrow[r, "F"'] & C
\end{tikzcd}$$

- The *family fibration* of a cartesian fibration $\pi : E \twoheadrightarrow B$, where B has all pullbacks:

$$\begin{tikzcd}
\pi \downarrow B \arrow[r] \arrow[d, two heads] \arrow[dd, two heads, bend right, "\mathrm{Fam}(\pi) :\equiv \partial_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "\pi"] \\
B^{\Delta^1} \arrow[r, "\partial_0"'] \arrow[d, two heads, "\partial_1"] & B \\
B &
\end{tikzcd}$$

Beck–Chevalley condition

Definition (Beck–Chevalley condition)

Let $P : B \to \mathcal{U}$ be a family over a Rezk type all of whose fibers are Rezk. Then P is said to satisfy the *Beck–Chevalley condition (BCC)* if for any dependent square of the form

$$\begin{tikzcd}
 & d' \arrow[r, "f'"] \arrow[d, "g'"'] & e' \arrow[d, "g"] \\
\widetilde{P} \arrow[dd, two heads] & d \arrow[r, "f"'] & e \\
 & a' \arrow[r, "u'"] \arrow[d, "v'"'] \arrow[dr, phantom, very near start, "\lrcorner"] & b' \arrow[d, "v"] \\
B & a \arrow[r, "u"'] & b
\end{tikzcd}$$

it holds that: if f is cocartesian, and g, g' are cartesian, then f' is cocartesian.

Beck–Chevalley families

Definition

A map between $\pi : E \twoheadrightarrow B$ is a *Beck–Chevalley fibration* or a *cartesian fibration with internal sums* if:

1. The map π is a bicartesian fibration, *i.e.* a cartesian and cocartesian fibration.
2. The map π satisfies the Beck–Chevalley condition.

- Classical motivation:
- $\mathrm{Fam}(\mathbb{C}) \to \mathbf{Set}$ with fibers $\mathbb{C}^I$ has internal sums if and only if $\mathbb{C}$ has small sums
- Bénabou's perspective generalizes this to an arbitrary base

Beck–Chevalley families: Characterizations

Theorem (Characterizations of BCC fibrations, cf. [Str22])

Let $P : B \to \mathcal{U}$ be a cartesian family over a Rezk type B which has all pullbacks, with unstraightening $\pi : E \twoheadrightarrow B$ be a cartesian fibration. Then π is Beck–Chevalley fibration if and only if the mediating fibered functor

$$\iota_P : E \to_B \pi \downarrow B, \; \iota_P(b, e) :\equiv \langle b, \mathrm{id}_b, e \rangle$$

has a fibered left adjoint which is also a cartesian functor:

$$\begin{tikzcd}
E \arrow[rr, bend right=10, "\iota_\pi"'] \arrow[rr, phantom, bend left=15, "\perp"] \arrow[dr, two heads, "\pi"'] & & \pi \downarrow B \arrow[ll, dashed, bend right=30, "\tau_\pi"'] \arrow[dl, two heads, "\partial_1"] \\
& B &
\end{tikzcd}$$

Internal sums for gluing

Proposition (Internal sums for gluing, [Str22])

Let A and B be Rezk types with pullbacks and $F : A \to B$ an arbitrary functor (hence all its fibers are Rezk). Then the following are equivalent:

1. *The functor F preserves pullbacks.*
2. *The gluing fibration $\mathrm{gl}(F) \equiv \partial_1 : B \downarrow F \twoheadrightarrow A$ is a Beck–Chevalley fibration.*

Extensive sums I

Recall from classical 1-category theory that a category $\mathbb{C}$ with pullbacks and coproducts is *extensive* if and only if, for all small families $(A_i)_{i \in I}$ the induced functor $\prod_{i \in I} \mathbb{C}/A_i \to \mathbb{C}/\coprod_{i \in I} A_i$ is an equivalence. This is equivalent to the condition that injections into finite sums are stable under pullback, and for any family of squares

$$\begin{tikzcd}
B_k \arrow[r, "g_k"] \arrow[d, "f_k"'] & B \arrow[d] \\
A_k \arrow[r] & \coprod_{i \in I} A_i
\end{tikzcd}$$

all of these are pullbacks if and only if all $g_k : B_k \to B$ are coproduct cones. This generalizes fibrationally as follows.

Extensive sums II

Definition (Stable and disjoint internal sums)

Let $P : B \to \mathcal{U}$ be a lex fibration with internal sums over a Rezk type B. Then P has *stable* internal sums if cocartesian arrows are stable under arbitrary pullbacks. The internal sums of P are *disjoint*[a] if for every cocartesian arrow $f : d \twoheadrightarrow^{P} e$ the fibered diagonal is cocartesian, too:

$$\begin{tikzcd}
d \arrow[dr, twoheadrightarrow, "\delta_f"] \arrow[drr, bend left, equal] \arrow[ddr, bend right, equal] & & \\
& d \times_e d \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & d \arrow[d, twoheadrightarrow, "f"] \\
& d \arrow[r, twoheadrightarrow, "f"'] & e
\end{tikzcd}$$

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[a] In a category, a coproduct is *disjoint* if the inclusion maps are monomorphisms, and the intersection of the summands is an initial object.

Extensive sums III

Definition ((Pre-)Moens families)

Let B be a lex Rezk type. A lex Beck–Chevalley family $P : B \to \mathcal{U}$ is a *pre-Moens family* if it has stable internal sums. We call a pre-Moens family $P : B \to \mathcal{U}$ *Moens family* (or *lextensive family* or *pre-geometric family*) if, moreover, all its (stable) internal sums are also disjoint.

When are internal sums extensive? I

Definition (Terminal elements)

Let B be a Segal type. An element $z : B$ is called *terminal* if $\mathrm{isContr}\left(\sum_{x:B} \hom_B(x,z)\right)$.

Definition (Choice of terminal elements)

Let B be a Rezk type and $P : B \to \mathcal{U}$ be a family with Rezk fibers such that every fiber has a terminal element. Then we denote, by type-theoretic choice, the section choosing fiberwise terminal elements by

$$\zeta_P :\equiv \zeta : \prod_{b:B} P\,b,$$

i.e. for any $b : B$ the element $\zeta_b : P\,b$ is terminal.

We define

$$\omega' \equiv \omega'_P : B \to P\,z, \quad \omega'(b) :\equiv \omega(\zeta_b) \equiv (!_b)_!(\zeta_b).$$

When are internal sums extensive? II

Proposition (Stable disjoint sums in terms of extensive sums, cf. [Str22])

Let B be a lex Rezk type and $P : B \to \mathcal{U}$ be a Beck–Chevalley family. Then, the following are equivalent:

- *The family P is a Moens family, i.e. P has stable disjoint sums.*
- *The bicartesian family P has internally extensive sums, i.e. for vertical arrows $f : d \to d'$, $k : e \to e'$, cocartesian arrows $g : e \xrightarrow{\ggg} e'$, in a square*

$$\begin{tikzcd} d \arrow[r, "h"] \arrow[d, squiggly, "f"'] & e \arrow[d, squiggly, "k"] \\ d' \arrow[r, "\ggg" marking, "g"'] & e' \end{tikzcd}$$

the arrow $h : d \to e$ is a cocartesian arrow if and only if the square is a pullback.

When are internal sums extensive? III

Proposition (cont'd)

- *The internal sums in P are Lawvere-extensive, i.e. in any square of the form*

$$\begin{tikzcd}
d \arrow[r, "h"] \arrow[d, squiggly, "!^a_d"'] & e \arrow[d, squiggly, "k"] \\
\zeta_a \arrow[r, tail reversed, "P_!(!_a, \zeta_a)"'] & \omega'(a)
\end{tikzcd}$$

  where $k : e \rightsquigarrow \omega'(a)$ is vertical the arrow $h : d \to e$ is cocartesian if and only if the given square is a pullback.

- *Let $z : B$ be a terminal element in B. For any $a : B$, the transport functor $(!_a)_! : P\,a \to P\,z$ reflects isomorphisms and $k^* P_!(!_a, \zeta_a)$ is cocartesian in case k is vertical.*

When are internal sums extensive? IV

Corollary

Let B be a Rezk type and $P : B \to \mathcal{U}$ a Moens family.
Then, for all arrows $u : a \to b$ in B and points $d : P\,a$, the functor

$$u_! \downarrow d : P\,a \downarrow d \to P\,b \downarrow u_! d$$

is an equivalence. In particular, for $P\,a \simeq P\,a \downarrow \zeta_a$ we have equivalences

$$u_! \downarrow \zeta_a : P\,a \downarrow \zeta_a \simeq P\,b \downarrow u_! \zeta_a, \quad (!_a)_! \downarrow \zeta_a : P\,a \downarrow \zeta_a \simeq P\,z \downarrow \omega'(a).$$

When are internal sums extensive? V

Corollary (Pullback preservation of covariant transport in Moens families)

Let B be a Rezk type and $P : B \to \mathcal{U}$ a Moens family.
Then for all $u : a \to b$ in B, the covariant transport functor

$$u_! : P\,a \to P\,b$$

preserves pullbacks.

Moens' Theorem for synthetic $(\infty, 1)$-categories I

We can adapt the proof from [Str22]:

Theorem (Moens' Theorem in simplicial HoTT (W, cf. [Str22]))

For a small lex Rezk type $B : \mathcal{U}$ the type

$$\mathrm{MoensFam}(B) :\equiv \sum_{P:B\to\mathcal{U}} \mathrm{isMoensFam}\, P$$

of $\mathcal{U}$-small Moens families is equivalent to the type

$$B \downarrow^{\mathrm{lex}} \mathrm{LexRezk} :\equiv \sum_{C:\mathrm{LexRezk}} (B \to^{\mathrm{lex}} C)$$

of lex functors from B into the type $\mathrm{LexRezk}$ of $\mathcal{U}$-small lex Rezk types.

Moens' Theorem for synthetic $(\infty, 1)$-categories II

Idea: Quasi-inverses given by "terminal transport" and gluing

$$\begin{tikzcd}
\mathrm{MoensFam}(B) \arrow[rr, bend left, "\Phi"] \arrow[rr, phantom, "\simeq"] & & B \downarrow^{\mathrm{lex}} \mathrm{LexRezk} \arrow[ll, bend left, "\Psi"]
\end{tikzcd}$$

i.e.

$$\Phi(P : B \to \mathcal{U}) \equiv \langle P\,z, \lambda b.(!_b)_!(\zeta_b) \rangle : B \to P\,z, \quad \Psi(F : B \to C) :\equiv \mathrm{St}_B\left(\mathrm{gl}(F) : C \downarrow F \twoheadrightarrow B\right) :$$

with $z : B$ terminal, and $\zeta : \Pi_B P$ picking the terminal element in each fiber:

$$\begin{tikzcd}
E \arrow[d, two heads, "\pi_P"'] & \zeta_b \arrow[rr, "\ggg"{marking}] & & \Phi_P(b) \\
B & b \arrow[rr, dashed, "!_b"] & & z
\end{tikzcd}$$

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Thank you for your attention!