On the logic of sets in the simplicial model

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Motivation

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However, verifying expected properties of the simplicial model is not always immediate.

For instance, while the proof that LEM holds in the said model [KL20] is not complicated per se, it is not as straightforward as one might expect.

The goal of today is to sketch a way to transport properties of the set model to the simplicial model with h-sets.

Properties of sets in the simplicial model: AC

For any $\Gamma \vdash A$ and $\Gamma, A \vdash B$ in the **Set** model, the type

$$\Gamma \vdash (\Pi_{a:A}\|B_a\|) \to \|\Pi_{a:A}B_a\|$$

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If we now take $\Gamma \vdash A$ and $\Gamma, A \vdash B$ in the **sSet** model with terms of

$$\Gamma \vdash \text{isSet}\, A \quad \text{and} \quad \Gamma, A \vdash \text{isSet}\, B,$$

is the corresponded type still inhabited?

Properties of sets in the simplicial model

Consider a proposition P constructed by

$$\frac{\Gamma_1 \vdash A_1 \text{ type} \quad \ldots \quad \Gamma_n \vdash A_n \text{ type} \qquad J_1 \quad \ldots \quad J_m}{\Gamma \vdash P \text{ type.}}$$

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If P is inhabited in the **Set** model, is the proposition

$$\frac{\begin{array}{ccc} \Gamma_1 \vdash A_1 \text{ type} & \Gamma_1 \vdash w_1 : \text{isSet}\, A_1 & J_1 \\ \vdots & \vdots & \vdots \\ \Gamma_n \vdash A_n \text{ type} & \Gamma_n \vdash w_n : \text{isSet}\, A_n & J_m \end{array}}{\Gamma \vdash P \text{ type.}}$$

inhabited as well in the **sSet** model?

Categories with Attributes

A category with attributes consists of

- a category $\mathbf{C}$ of contexts,
- a presheaf $\mathrm{Ty} : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ of types,
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Example: the **Set** model

$$\mathbf{C} = \mathbf{Set} \quad \text{and} \quad \mathrm{Ty}(J) = \mathrm{Fam}(J) = \mathbf{Set}^J$$

Sets over simplicial sets

In the simplicial model, **C** is **sSet** and $\mathrm{Ty}(\Gamma)$ is essentially the set of Kan fibrations with codomain Γ, which we write $\mathrm{Fib}(\Gamma)$.

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There are two different CwA structures on **sSet** worth considering:

- The CwA of h-sets where $\mathrm{Ty}(\Gamma) = \mathrm{Fib}_0(\Gamma) = (A : \mathrm{Fib}(\Gamma)) \times \mathrm{isSet}(A)$.
- The CwA of discrete sets where $\mathrm{Ty}(\Gamma) = \mathrm{DFib}(\Gamma) = \mathbf{sSet}(\Gamma, \mathcal{N}\,\mathbf{Set}_{\cong})$.

Another way to see discrete sets is as a generalisation of Fam, because

$$\mathbf{sSet}(\Gamma, \mathcal{N}\,\mathbf{Set}_{\cong}) \cong \mathbf{Cat}(\tau\Gamma, \mathbf{Set}_{\cong}),$$

where $\tau : \mathbf{sSet} \to \mathbf{Cat}$ is the associated category functor.

From sets to simplicial sets

We can break down the evident CwA map

$$(\mathbf{Set}, \mathrm{Fam}) \longrightarrow (\mathbf{sSet}, \mathrm{Fib})$$

as factors

$$(\mathbf{Set}, \mathrm{Fam}) \longrightarrow (\mathbf{sSet}, \mathrm{DFib}) \longrightarrow (\mathbf{sSet}, \mathrm{Fib}_0) \longrightarrow (\mathbf{sSet}, \mathrm{Fib})$$

where

$$\begin{aligned} \mathrm{Fam}(J) &= \mathbf{Set}^J \\ \mathrm{DFib}(\Gamma) &= \mathbf{sSet}(\Gamma, \mathcal{N}\,\mathbf{Set}_{\cong}) \\ \mathrm{Fib}_0(\Gamma) &= (A : \mathrm{Fib}(\Gamma)) \times \mathrm{isSet}(A) \\ \mathrm{Fib}(\Gamma) &= \{\text{Kan fibrations over } \Gamma\} \end{aligned}$$

Local equivalence of CwAs

A CwA map $F : \mathbf{C} \to \mathbf{D}$ is a *local equivalence* [KL18] if it satisfies:

- *Weak type lifting*: for any $\Gamma \in \mathbf{C}$ and $A \in \mathrm{Ty}_{\mathbf{D}}(F\Gamma)$, there exists $\overline{A} \in \mathrm{Ty}_{\mathbf{C}}(\Gamma)$ together with an equivalence $F\overline{A} \xrightarrow{\sim} A$ over $F\Gamma$.

- *Weak term lifting*: for any $\Gamma \in \mathbf{C}$, $A \in \mathrm{Ty}_{\mathbf{C}}(\Gamma)$, and $a \in \mathrm{Tm}(FA)$, there exists $\overline{a} \in \mathrm{Tm}(A)$ together with an element of the identity type $\mathrm{Id}_{FA}(F\overline{a}, a)$.

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When a CwA map is a local equivalence it also satisfies *weak context lifting* and *weak section lifting* (the iterated versions of the above).

Local isomorphism of CwAs

A CwA map $F : \mathbf{C} \to \mathbf{D}$ is a *local isomorphism* if it satisfies:

- *Type lifting*: for any $\Gamma \in \mathbf{C}$ and $A \in \mathrm{Ty}_{\mathbf{D}}(F\Gamma)$, there exists $\overline{A} \in \mathrm{Ty}_{\mathbf{C}}(\Gamma)$ together with an isomorphism $F\overline{A} \xrightarrow{\cong} A$ over $F\Gamma$.
- *Term lifting*: for any $\Gamma \in \mathbf{C}$, $A \in \mathrm{Ty}_{\mathbf{C}}(\Gamma)$, and $a \in \mathrm{Tm}(FA)$, there exists $\overline{a} \in \mathrm{Tm}(A)$ such that $F(\overline{a}) = a$ (up to $F(\Gamma.A) \cong F\Gamma.FA$).

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The map $(\mathbf{Set}, \mathrm{Fam}) \to (\mathbf{sSet}, \mathrm{DFib})$ is a local isomorphism: If $J \in \mathbf{Set}$, then

$$A \in \mathrm{DFib}(J) = \mathbf{sSet}(J, \mathcal{N}\,\mathbf{Set}_{\cong}) \cong \mathbf{Cat}(\tau J, \mathbf{Set}_{\cong})$$

is induced by a J-family of sets because τJ is discrete.

$(\mathbf{sSet}, \mathrm{DFib}) \to (\mathbf{sSet}, \mathrm{Fib}_0)$ is a local equivalence

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From $A \in \mathrm{Fib}_0(\Gamma)$, need $\overline{A} \in \mathrm{DFib}(\Gamma)$ equivalent over Γ.
In terms of $\tau\Gamma \to \mathbf{Set}_{\cong}$, we can take

$$\overline{A}_v = \pi_0(A_v) \quad \text{for } v \in \Gamma_0$$

and if f is a 1-simplex of Γ from v to w, the 1-simplices of A above f induce a map $\pi(A_v) \to \pi(A_w)$.

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In this way, the fibrations A and $\overline{A}$ are fibrewise equivalent, hence equivalent over Γ.

Type formers: Propositional truncation

What about type formers? It is important that they play well with

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As a guiding example, we only consider the case of propositional truncation $\|-\|$ today.

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In the set model, for $A \in \mathrm{Fam}(J)$, this is modelled by

$$(\|A\|)_j = \begin{cases} \{*\} & \text{if } A_j \neq \emptyset, \\ \emptyset & \text{if } A_j = \emptyset. \end{cases}$$

And in the simplicial model, it is obtained as a special case of more general methods for interpreting higher inductive types [LS20].

Truncation via image factorisations

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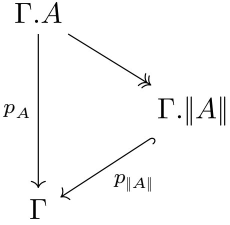
The natural candidate is image factorisation:



Coherence with substitutions

As expected, this naive definition will not commute with substitutions on the nose, but the usual trick of defining it via a universe works.

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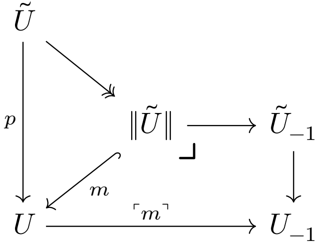

and for $A \in \mathrm{Fib}\,\Gamma = \mathbf{sSet}(\Gamma, U)$ define $\|A\|$ as

$$\Gamma \xrightarrow{A} U \xrightarrow{\ulcorner m \urcorner} U_{-1} \to U$$

Preservation and computation

Since the map $\mathbf{Set} \to \mathbf{sSet}$ preserves epi-mono factorisations, the CwA map

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preserves propositional truncation up to isomorphism.
The same is true of the intermediate maps

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Remark: with this interpretation of propositional truncation, β-reduction does not hold definitionally, but overall this is still equivalent to the more homotopical definition of truncation.

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The above only yields preservation for contexts in the image of $\mathbf{Set} \to \mathbf{sSet}$, we can do better.

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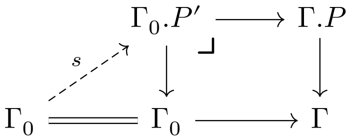
Given a context $\Gamma \in$ **sSet** and a proposition P over it, we pull it back



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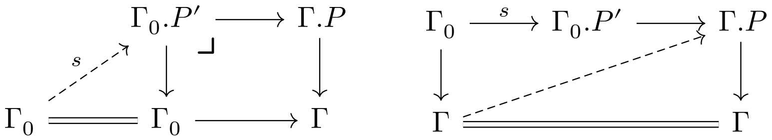

and we get a term of P from a term s of P' because $\Gamma_0 \to \Gamma$ is (-1)-connected and $\Gamma.P \to \Gamma$ is (-1)-truncated.

Thank you!

[KL20] Chris Kapulkin and Peter Lefanu Lumsdaine. "The law of excluded middle in the simplicial model of type theory". English. *Theory Appl. Categ.* 35 (2020), pp. 1546–1548.

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[LS20] Peter Lefanu Lumsdaine and Michael Shulman. "Semantics of higher inductive types". *Mathematical Proceedings of the Cambridge Philosophical Society* 169.1 (2020), pp. 159–208.